

FUNGSI KHUSUS
DALAM BENTUK
INTEGRAL

FUNGSI FAKTORIAL

Definisi

$$\int_0^{\infty} x^n e^{-x} dx = n!$$

Buktikan bahwa : **$0! = 1$**

$$0! = \int_0^{\infty} x^0 e^{-x} dx = \int_0^{\infty} e^{-x} dx = -e^{-x} \Big|_0^{\infty} = -(0-1) = 1$$

Terbukti

FUNGSI Gamma

Definisi

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} dx; \quad p > 0$$

Hubungan fungsi Gamma dengan fungsi Faktorial

$$\Gamma(p+1) = \int_0^{\infty} x^{(p-1)+1} e^{-x} dx = \int_0^{\infty} x^p e^{-x} dx = p!$$

$$\Gamma(p+1) = p!$$

$$\Gamma(1) = 0! = 1, \quad \Gamma(2) = 1! = 1, \quad \Gamma(3) = 2! = 2 \quad dst$$

Nilai Fungsi Gamma ditabulasi
untuk Gamma 1 sampai
dengan Gamma 2

Hubungan Rekursif Fungsi Gamma

$$\Gamma(p+1) = \int_0^{\infty} x^p e^{-x} dx$$

Lakukan integrasi parsial, seperti berikut :

$$u = x^p \quad du = px^{p-1} dx$$

$$dv = e^{-x} dx \quad v = -e^{-x}$$

$$\Gamma(p+1) = -x^p e^{-x} \Big|_0^{\infty} - \int_0^{\infty} (-e^{-x}) px^{p-1} dx$$

$$\Gamma(p+1) = p \int_0^{\infty} x^{p-1} e^{-x} dx = p\Gamma(p)$$

$$\boxed{\Gamma(p+1) = p\Gamma(p)}$$

Hubungan Rekursif Fungsi Gamma

$$\Gamma(p+1) = p\Gamma(p)$$

$$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2(1) = 2$$

$$\Gamma(4) = \Gamma(3+1) = 3\Gamma(3) = 3\Gamma(2+1) = 3 \cdot 2\Gamma(2) = 6(1) = 6$$

dst

Hubungan Rekursif Fungsi Gamma

$$\Gamma(p) = \frac{1}{p} \Gamma(p+1)$$

$$\Gamma(0,6) = \frac{1}{0,6} \Gamma(0,6+1) = \frac{1}{0,6} \Gamma(1,6) \longrightarrow \text{Tabel F. Gamma}$$

$$\Gamma(-1,5) = \frac{1}{-1,5} \Gamma(-1,5+1) = \frac{1}{-1,5} \Gamma(-0,5)$$

$$= \frac{1}{-1,5} \frac{1}{-0,5} (-0,5+1)$$

$$= \frac{1}{-1,5} \frac{1}{-0,5} \Gamma(0,5)$$

$$= \frac{1}{-1,5} \frac{1}{-0,5} \frac{1}{0,5} \Gamma(1,5) \longrightarrow \text{Tabel F. Gamma}$$

Nilai $\Gamma(0,5)$

$$\Gamma(0,5) = \int_0^{\infty} t^{0,5-1} e^{-t} dt = \int_0^{\infty} \frac{1}{\sqrt{t}} e^{-t} dt$$

Misalkan $t = y^2$, maka $dt = 2y dy$

$$\Gamma(0,5) = 2 \int_0^{\infty} \frac{1}{y} e^{-y^2} y dy$$

$$\Gamma(0,5) = 2 \int_0^{\infty} \frac{1}{x} e^{-x^2} x dx$$

$$[\Gamma(0,5)]^2 = 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy$$

Nilai $\Gamma(0,5)$

$$[\Gamma(0,5)]^2 = 4 \int_{\theta=0}^{\pi/2} \int_0^{\infty} e^{-r^2} r dr d\theta = 4 \frac{\pi}{2} \frac{e^{-r^2}}{-2} \Big|_0^{\infty} = \pi$$

$$[\Gamma(0,5)]^2 = \pi$$

$$\Gamma(0,5) = \sqrt{\pi}$$

Contoh soal

Hitunglah integral berikut dengan Fungsi Gamma :

$$\int_0^{\infty} x^2 e^{-x^2} dx$$

Jawab

$$\int_0^{\infty} x^2 e^{-x^2} dx$$

Misal $u = x^2$, maka $du = 2x dx$

Untuk $x = 0$ maka $u = 0$

Untuk $x = \infty$ maka $u = \infty$

$$\int_0^{\infty} u e^{-u} \frac{du}{2\sqrt{u}} = \frac{1}{2} \int_0^{\infty} u^{1/2} e^{-u} du = \frac{1}{2} \Gamma(1,5) = \frac{1}{2} \frac{1}{2} \Gamma(0,5) = \frac{\sqrt{\pi}}{4}$$

$$\int_0^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}$$

Soal Latihan

Hitunglah integral-integral berikut dengan Fungsi Gamma :

$$1. \int_0^{\infty} \sqrt{t} e^{-3t} dt$$

$$2. \int_0^{\infty} (2y^2 + y) e^{-y^2} dy$$

$$3. \int_0^1 (\ln x)^{1/3} dx$$

Formula penting terkait Fungsi Gamma

$$\Gamma(p) \Gamma(1-p) = \frac{\pi}{\sin \pi p}$$

Untuk $p = 0,5$

$$\Gamma(0,5) \Gamma(1-0,5) = \frac{\pi}{\sin 0,5\pi}$$

$$\Gamma(0,5) \Gamma(0,5) = \frac{\pi}{1}$$

$$[\Gamma(0,5)]^2 = \pi$$

$$\Gamma(0,5) = \sqrt{\pi}$$

Contoh soal

Buktikan bahwa :

$$z!(-z)! = \frac{\pi z}{\sin \pi z}$$

Bukti

$$z!(-z)! = \Gamma(z+1) \Gamma(1-z)$$

$$z!(-z)! = z\Gamma(z) \Gamma(1-z)$$

$$z!(-z)! = z[\Gamma(z)\Gamma(1-z)]$$

$$z!(-z)! = z \left[\frac{\pi}{\sin \pi z} \right]$$

$$z!(-z)! = \frac{\pi z}{\sin \pi z}$$

Ingat :

$$\Gamma(n+1) = n!$$

dan

$$\Gamma(n+1) = n\Gamma(n)$$

dan

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin \pi n}$$

Fungsi Beta

Definisi 1 :

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx; \quad p > 0; q > 0.$$

$$B(p, q) = B(q, p)$$

Definisi 2 :

$$B(p, q) = \frac{1}{a^{p+q-1}} \int_0^a y^{p-1} (a-y)^{q-1} dy$$

Fungsi Beta

Definisi 3 :

$$B(p, q) = 2 \int_0^{\pi/2} (\sin \theta)^{2p-1} (\cos \theta)^{2q-1} d\theta$$

Definisi 4 :

$$B(p, q) = \int_0^{\infty} \frac{y^{p-1} dy}{(1+y)^{p+q}}$$

Hubungan Fungsi Beta dan Fungsi Gamma

Fungsi Gamma

$$\Gamma(p) = \int_0^{\infty} t^{p-1} e^{-t} dt$$

Misal $t = y^2$, maka

$$\Gamma(p) = 2 \int_0^{\infty} y^{2p-1} e^{-y^2} dy \quad \text{atau} \quad \Gamma(q) = 2 \int_0^{\infty} x^{2q-1} e^{-x^2} dx$$

Jika $\Gamma(p)$ dikalikan dengan $\Gamma(q)$ dikalikan maka :

$$\Gamma(p)\Gamma(q) = 4 \int_0^{\infty} \int_0^{\infty} x^{2q-1} y^{2p-1} e^{-(x^2+y^2)} dx dy$$

$$\Gamma(p)\Gamma(q) = 4 \int_0^{\infty} \int_0^{\pi/2} (r \cos \theta)^{2q-1} (r \sin \theta)^{2p-1} e^{-r^2} r dr d\theta$$

Hubungan Fungsi Beta dan Fungsi Gamma

$$\Gamma(p)\Gamma(q) = 4 \int_0^{\infty} r^{2p+2q-1} e^{-r^2} dr \int_0^{\pi/2} (\cos \theta)^{2q-1} (\sin \theta)^{2p-1} d\theta$$

$$\frac{1}{2} \Gamma(p+q)$$

$$\frac{1}{2} B(p, q)$$

$$\Gamma(p)\Gamma(q) = 4 \frac{1}{2} \Gamma(p+q) \cdot \frac{1}{2} B(p, q)$$

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Hubungan Fungsi Beta dan Fungsi Gamma

contoh

$$B\left(1, \frac{1}{2}\right) = \frac{\Gamma(1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{(1)(\sqrt{\pi})}{\frac{1}{2}\sqrt{\pi}} = 2$$

Contoh soal 1

Selesaikan integral berikut dengan fungsi Beta

$$\int_0^{\pi/2} \sqrt{\sin^3 x \cos x} dx$$

Solusi dengan def. Fungsi Beta 3

$$B(p, q) = 2 \int_0^{\pi/2} (\sin \theta)^{2p-1} (\cos \theta)^{2q-1} d\theta$$

$$\int_0^{\pi/2} (\sin^3 x \cos x)^{1/2} dx = \int_0^{\pi/2} (\sin x)^{3/2} (\cos x)^{1/2} dx$$

$$2p-1 = \frac{3}{2} \quad p = \frac{5}{4} \quad 2q-1 = \frac{1}{2} \quad q = \frac{3}{4}$$

$$\int_0^{\pi/2} \sqrt{\sin^3 x \cos x} dx = \frac{1}{2} B\left(\frac{5}{4}, \frac{3}{4}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(2)} = \dots$$

Contoh soal 2

Selesaikan integral berikut dengan fungsi Beta

$$\int_0^{\infty} \frac{y^2 dy}{(1+y)^6}$$

Solusi dengan def. Fungsi Beta 4

$$B(p, q) = \int_0^{\infty} \frac{y^{p-1} dy}{(1+y)^{p+q}}$$

$$p-1=2 \quad p=3 \quad (p+q)=6 \quad q=3$$

Jadi

$$\int_0^{\infty} \frac{y^2 dy}{(1+y)^6} = B(3,3) = \frac{\Gamma(3)\Gamma(3)}{\Gamma(6)} = \frac{4}{120}$$

Soal Latihan

Selesaikan integral berikut dengan fungsi Beta yang sesuai

$$1. \int_0^{\infty} \frac{y \, dy}{(1 + y^3)^2}$$

$$2. \int_0^2 \frac{x^2 \, dx}{\sqrt{2-x}}$$

$$3. \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta}}$$

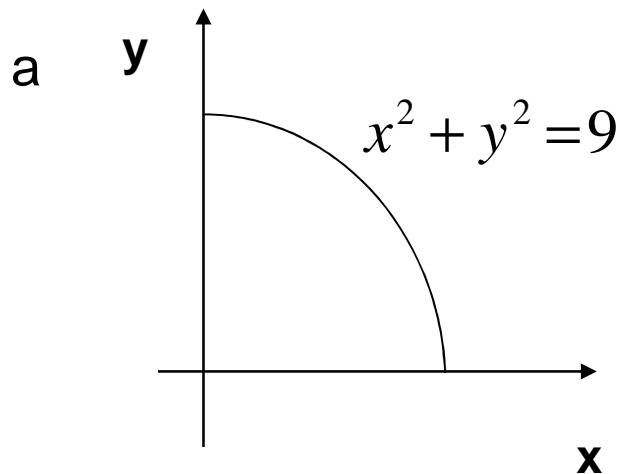
Aplikasi dalam persoalan Fisika

Sket grafik $x^2 + y^2 = 9$

Gunakan fungsi Beta untuk menghitung :

- Luas daerah pada kuadran pertama
- Titik pusat massa dari daerah ini (anggap rapat massanya seragam)

Jawab



$$A = \int dA = \iint dx dy = \int_0^3 \int_0^{\sqrt{9-y^2}} dx dy$$

$$A = \int_0^3 \sqrt{9-y^2} dy$$

Aplikasi dalam persoalan Fisika

$$A = \int_0^3 \sqrt{9 - y^2} \, dy \quad \text{Misalkan : } y^2 = x \longrightarrow 2y \, dy = dx$$

$$\text{Batas : } y=0 \longrightarrow x=0$$

$$y=3 \longrightarrow x=9$$

$$A = \int_0^9 \sqrt{9-x} \frac{dx}{2\sqrt{x}} = \frac{1}{2} \int_0^9 x^{-1/2} (9-x)^{1/2} dx$$

Gunakan Fungsi
Beta kedua

$$B(p, q) = \frac{1}{a^{p+q-1}} \int_0^a y^{p-1} (a-y)^{q-1} dy$$

$$\int_0^a y^{p-1} (a-y)^{q-1} dy = a^{p+q-1} B(p, q)$$

$$A = \frac{1}{2} (9)^{1/2+3/2-1} B(1/2, 3/2)$$

$$A = \frac{1}{2} (9) \frac{\Gamma(1/2)\Gamma(3/2)}{\Gamma(2)}$$

$$A = \frac{1}{2} (9) \frac{\sqrt{\pi} \cdot 1/2 \sqrt{\pi}}{1} = \frac{9\pi}{4}$$

Dengan rumus luas lingkaran :

$$A = \frac{1}{4} \pi r^2 = \frac{1}{4} \pi (3)^2 = \frac{9\pi}{4}$$

Sama

B. Titik pusat massa luasan tersebut dihitung dengan rumus :

$$x_{pm} = \frac{\int x dM}{\int dM} = \frac{\int \rho x dA}{\int \rho dA}$$

$$y_{pm} = \frac{\int y dM}{\int dM} = \frac{\int \rho y dA}{\int \rho dA}$$

Dst

Fungsi Error dan Pelengkapnya

Definisi Fungsi Error

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Definisi Pelengkap Fungsi Error

$$\operatorname{Erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$$

Fungsi Error

$$\operatorname{Erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt$$

Selesaikan dengan
Fungsi Gamma

$$\operatorname{Erf}(\infty) = \frac{2}{\sqrt{\pi}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

Fungsi Error dan Pelengkapnya

$$\operatorname{Erf}(x) + \operatorname{Erfc}(x) = \frac{2}{\sqrt{\pi}} \left(\int_0^x e^{-t^2} dt + \int_x^{\infty} e^{-t^2} dt \right)$$

$$\operatorname{Erf}(x) + \operatorname{Erfc}(x) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt$$

$$\operatorname{Erf}(x) + \operatorname{Erfc}(x) = \operatorname{Erf}(\infty) = 1$$

$$\operatorname{Erf}(x) = 1 - \operatorname{Erfc}(x)$$

$$\operatorname{Erfc}(x) = 1 - \operatorname{Erf}(x)$$

Fungsi Error

Dari deret pangkat tak hingga :

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots \rightarrow e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} + \dots$$

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \dots \right) dt$$

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \left(t - \frac{t^3}{3} + \frac{t^5}{5 \cdot 2!} - \dots \right)_0^x$$

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \dots \right) \quad |x| \ll 1$$

Pelengkap Fungsi Error

Definisi pelengkap fungsi error

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt.$$

Kita tuliskan :

$$e^{-t^2} = \frac{1}{t} te^{-t^2} = \frac{1}{t} \frac{d}{dt} \left(-\frac{1}{2} e^{-t^2} \right)$$

dan lakukan integrasi by part sbb :

Pelengkap Fungsi Error

$$\begin{aligned}\int_x^\infty e^{-t^2} dt &= \int_x^\infty \frac{1}{t} \frac{d}{dt} \left(-\frac{1}{2} e^{-t^2} \right) dt \\ &= \frac{1}{t} \left(-\frac{1}{2} e^{-t^2} \right) \Big|_x^\infty - \int_x^\infty \left(-\frac{1}{2} e^{-t^2} \right) \left(-\frac{1}{t^2} \right) dt \\ &= \frac{1}{2x} e^{-x^2} - \frac{1}{2} \int_x^\infty \frac{1}{t^2} e^{-t^2} dt.\end{aligned}$$

Sekarang tuliskan lagi :

$$\frac{1}{t^2} e^{-t^2} = \left(\frac{1}{t^3} \right) \left(\frac{d}{dt} \right) \left(-\frac{1}{2} e^{-t^2} \right)$$

kemudian lakukan integral by part lagi sbb :

Pelengkap Fungsi Error

$$\begin{aligned}\int_x^\infty \frac{1}{t^2} e^{-t^2} dt &= \int_x^\infty \frac{1}{t^3} \frac{d}{dt} \left(-\frac{1}{2} e^{-t^2} \right) dt \\ &= \frac{1}{t^3} \left(-\frac{1}{2} e^{-t^2} \right) \Big|_x^\infty - \int_x^\infty \left(-\frac{1}{2} e^{-t^2} \right) \left(-\frac{3}{t^4} \right) dt \\ &= \frac{1}{2x^3} e^{-x^2} - \frac{3}{2} \int_x^\infty \frac{1}{t^4} e^{-t^2} dt.\end{aligned}$$

Jika proses ini terus dilanjutkan, maka akan didapat ungkapan deret untuk pelengkap fungsi error, sbb :

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{(2x^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2x^2)^3} + \dots \right).$$

$$|x| \gg 1$$

Contoh soal

$$1. \int_0^2 e^{-x^2} dx$$

$$2. \frac{2}{\sqrt{\pi}} \int_5^{10} e^{-u^2} du$$

$$3. \sqrt{\frac{2}{\pi}} \int_1^{\infty} e^{-t^2/2} dt$$

Formula Stirling

Formula Stirling adalah formula pendekatan untuk fungsi Faktorial dan Fungsi Gamma, sbb :

$$n! \sim n^n e^{-n} \sqrt{2\pi n} \quad \text{or} \quad \Gamma(p+1) \sim p^p e^{-p} \sqrt{2\pi p}.$$

Bukti

$$\Gamma(p+1) = p! = \int_0^{\infty} x^p e^{-x} dx = \int_0^{\infty} e^{p \ln x - x} dx.$$

Substitusi variabel baru y sehingga $x = p + y\sqrt{p}.$

$$dx = \sqrt{p} dy,$$

Formula Stirling

persamaan di atas menjadi :

$$p! = \int_{-\sqrt{p}}^{\infty} e^{p \ln(p + y\sqrt{p}) - p - y\sqrt{p}} \sqrt{p} dy.$$

untuk p besar, logaritma dapat diekspansi dalam deret pangkat berikut :

$$\ln(p + y\sqrt{p}) = \ln p + \ln\left(1 + \frac{y}{\sqrt{p}}\right) = \ln p + \frac{y}{\sqrt{p}} - \frac{y^2}{2p} + \dots$$

sehingga

Formula Stirling

$$p! \sim \int_{-\sqrt{p}}^{\infty} e^{p \ln p + y\sqrt{p} - (y^2/2) - p - y\sqrt{p}} \sqrt{p} dy$$

$$= e^{p \ln p - p} \sqrt{p} \int_{-\sqrt{p}}^{\infty} e^{-y^2/2} dy$$

$$= p^p e^{-p} \sqrt{p} \left[\int_{-\infty}^{\infty} e^{-y^2/2} dy - \int_{-\infty}^{-\sqrt{p}} e^{-y^2/2} dy \right].$$

$\sqrt{2\pi}$

0

Untuk p besar

$$p! \sim p^p e^{-p} \sqrt{2\pi p}$$

Bandingkan nilai eksak $n!$ dengan formula pendekatan Stirling

n	$n!$ eksak	$n!$ Formula Stirling	Persen selisih
5	120	118,0	1,7 %
20	$2,43 \times 10^{18}$	$2,42 \times 10^{18}$	1,0 %
50	$3,04 \times 10^{64}$		

Soal

1. Dalam mekanika statistik sering digunakan persamaan :

$$\ln N! = N \ln N - N$$

disini N berorde bilangan Avogadro, $N = 10^{26}$

Buktikan dari Formula Stirling !

2. Hitunglah :

$$\lim_{n \rightarrow \infty} \frac{(2n)! \sqrt{n}}{2^{2n} (n!)^2}$$

3. Hitunglah : $\Gamma(55,5)$

Integral Eliptik

Bentuk Legendre

disebut juga integral eliptik tak lengkap jenis ke satu dan ke dua

$$F(k, \phi) = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad \left\{ \begin{array}{l} 0 \leq k \leq 1, \\ \text{or} \\ k = \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}. \end{array} \right.$$
$$E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \phi} d\phi,$$

k disebut modulus dan ϕ disebut amplitudo integral eliptik. Integral ini ditabulasi untuk nilai $\theta = \arcsin k$ dan ϕ antara 0 dan $\pi/2$.

k^2 dapat dilihat dari bentuk integral, dengan mengetahui k maka θ dapat ditentukan, sedangkan ϕ dapat dilihat pada batas integral, dengan mengetahui θ dan ϕ , maka nilai integral eliptik dapat dilihat pada tabel integral eliptik $F(k, \phi)$ dan $E(k, \phi)$.

Integral Eliptik

Integral eliptik lengkap

Integral eliptik lengkap jenis pertama dan kedua adalah nilai-nilai K dan E (sebagai fungsi k) untuk $\phi = \pi/2$

$$K \quad \text{or} \quad K(k) = F\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

$$E \quad \text{or} \quad E(k) = E\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \phi} \, d\phi.$$

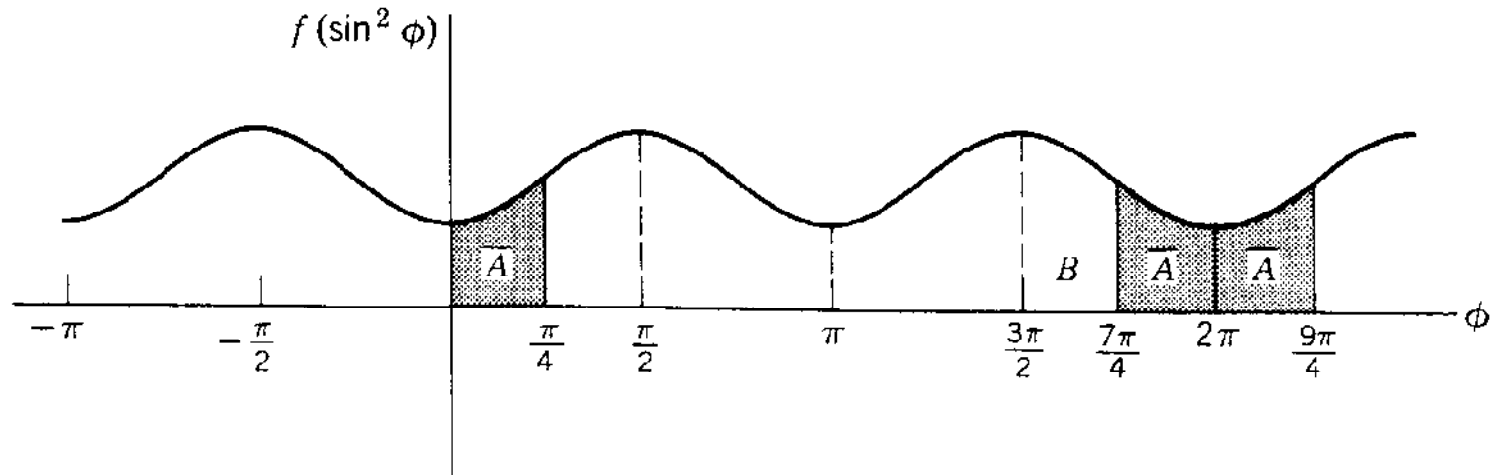
Sama seperti sebelumnya, k^2 dapat dilihat dari bentuk integral, dengan mengetahui k maka θ dapat ditentukan, dengan mengetahui θ , maka nilai integral eliptik lengkap dapat dilihat pada tabel integral eliptik lengkap K dan E

Contoh soal

$$1. \int_0^{\pi/4} \frac{d\phi}{\sqrt{1-0,25 \sin^2 \phi}}$$

Bagaimana menghitung integral eliptik untuk $\phi > \pi/2$???

Tinjau fungsi $\sin^2 x$ [$f(\sin^2 x)$] yang merupakan integran dari integral eliptik



$$\int_0^{9\pi/4} \dots = \int_0^{2\pi} \dots + \text{luas } A = \int_0^{2\pi} \dots + \int_0^{\pi/4} \dots = 4 \int_0^{\pi/2} \dots + \int_0^{\pi/4} \dots$$

$$\int_0^{7\pi/4} \dots = \int_0^{2\pi} \dots - \text{luas } A = \int_0^{2\pi} \dots - \int_0^{\pi/4} \dots = 4 \int_0^{\pi/2} \dots - \int_0^{\pi/4} \dots$$

catat

$$\int_0^{7\pi/4} \dots \neq \int_0^{3\pi/2} \dots + \text{luas } A = \int_0^{3\pi/2} \dots + \int_0^{\pi/4} \dots = 3 \int_0^{\pi/2} \dots + \int_0^{\pi/4} \dots$$

Contoh soal

$$\int_0^{5\pi/4} \sqrt{1 - 0,037 \sin^2 \phi} \, d\phi$$

Jika batas bawah integral tidak nol, maka :

$$\begin{aligned}\int_{\phi_1}^{\phi_2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} &= \int_0^{\phi_2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} - \int_0^{\phi_1} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \\ &= F(k, \phi_2) - F(k, \phi_1),\end{aligned}$$

dan jika salah satu batas integral adalah negatif, maka :

$$F(k, -\phi) = \int_0^{-\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = - \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = -F(k, \phi)$$

Karena $F(k, \phi)$ dan $E(k, \phi)$ merupakan fungsi ganjil

Contoh soal

$$\int_{-7\pi/8}^{11\pi/4} \sqrt{1 - 0,64 \sin^2 \phi} \, d\phi$$

Bentuk Jacobi

Jika kita ambil $x = \sin \phi$, pada bentuk Legendre, maka akan didapat integral eliptik bentuk Jacobi jenis pertama dan kedua, sbb :

$$x = \sin \phi,$$

$$dx = \cos \phi d\phi \quad \text{or} \quad d\phi = \frac{dx}{\cos \phi} = \frac{dx}{\sqrt{1-x^2}},$$

$$\phi = \pi/2 \text{ corresponds to } x = 1.$$

dan

$$F(k, \phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \int_0^x \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}},$$

$$E(k, \phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \phi} d\phi = \int_0^x \sqrt{\frac{1 - k^2 x^2}{1 - x^2}} dx,$$

$$K = F\left(k, \frac{\pi}{2}\right) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}},$$

$$E = \int_0^1 \sqrt{\frac{1 - k^2 x^2}{1 - x^2}} dx.$$

Contoh soal

$$\int_0^{0,8} \frac{dx}{\sqrt{(1-x^2)(1-0,16x^2)}}$$

Contoh soal

$$1. \int_0^{5\pi/4} \sqrt{1 - 0,037 \sin^2 \phi} \, d\phi$$

$$3. \int_{-7\pi/8}^{11\pi/4} \sqrt{1 - 0,64 \sin^2 \phi} \, d\phi$$

$$2. \int_0^{0,5} \sqrt{\frac{100 - x^2}{1 - x^2}} \, dx$$

$$4. \int_{-0,5}^{0,5} \frac{dx}{\sqrt{(1 - x^2)(4 - 3x^2)}}$$