CHAPTER II
UNIVERSAL GRAVITATION

Outline:

⊙ *Newton's Law of Universal Gravitation*
- *Measuring gravitational constant*
- *Free-fall acceleration and the gravitational force*

⊙ *Energy Considerations in Planetary and Satellite Motion*
- *Orbital energy*
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⊙ *Tidal Force*
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⊙ *Kepler's Laws and the Motion of Planets*
- *Kepler's first law, second law and third law*

II.1 Newton's Law of Universal Gravitation

In 1687 Newton published his work on the law of gravity in his treatise *Mathematical Principles of Natural Philosophy.* Newton's law of universal gravitation states that *"every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them"*.

If the particles have masses m_1 and m_2 and are separated by a distance r, the magnitude of this gravitational force is

$$\mathbf{F_g = G\frac{m_1 m_2}{r^2}} \quad \text{.........}(2.1)$$

where G is a constant, called the *universal gravitational constant,* that has been measured experimentally. Its value in SI units is G = 6.67x10^{-11} Nm^{2}kg^{-2}. The form of the force law given by equation (2.1) is often referred to as an inversesquare law because the magnitude of the force varies as the inverse square of the separation of the particles.

In formulating his law of universal gravitation, Newton used the following reasoning, which supports the assumption that the gravitational force is proportional to the inverse square of the separation between the two interacting objects. He compared the acceleration of the Moon in its orbit with the acceleration of an object falling near the Earth's surface, such as the legendary apple (see **Figure 2.1**). Assuming that both accelerations had the same cause—namely, the gravitational attraction of the Earth—Newton used the inverse-square law to reason that the acceleration of the Moon toward the Earth (centripetal acceleration) should be proportional to $\mathbf{1/r_M^2}$, where r_M is the distance between the centres of the Earth and the Moon. Furthermore, the acceleration of the apple toward the Earth should be proportional to $\mathbf{1/R_a^2}$, where R_a is the distance between the centres of the Earth and the apple. Because the apple is located at the surface of the Earth, $R_a = R_E$, the radius of the Earth. Using the values r_M = 3.84x10^8 m and R_E = 6.37x10^6 m, Newton predicted that the ratio of the Moon's acceleration a_M to the apple's acceleration g would be

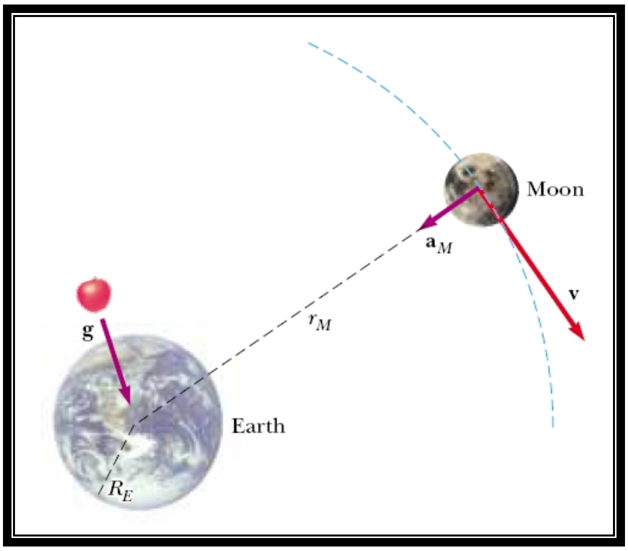

Figure 2.1 As it revolves around the Earth, the Moon experiences a centripetal acceleration $\mathbf{a}_M$ directed toward the Earth. An object near the Earth's surface, such as the apple shown here, experiences an acceleration **g**. (Dimensions are not to scale.)

$$\frac{a_M}{g} = \frac{\left(1/r_M\right)^2}{\left(1/R_E\right)^2} = \left(\frac{R_E}{r_M}\right)^2 \quad \ldots\ldots\ldots.(2.2)$$

If we substitute for R_E and r_M with their value, giving us $2.75\text{x}10^{-4}$. Therefore, the centripetal acceleration of the Moon, a_M, is: $(2.75\text{x}10^{-4})(9.80\ \text{ms}^{-2}) = 2.70\text{x}10^{-3}\ \text{ms}^{-2}$.

Newton also calculated the centripetal acceleration of the Moon from a knowledge of its mean distance from the Earth and the known value of its orbital period, T = 27.32 days = $2.36\text{x}10^6$ s. In a time interval T, the Moon travels a distance $2\pi r_M$, which equals the circumference of its orbit. Therefore, its orbital speed is $2\pi r_M/T$ and its centripetal acceleration is

$$a_M = \frac{v^2}{r_M} = \frac{\left(2\pi r_M/T\right)^2}{r_M} = \frac{4\pi^2 r_M}{T^2} \quad \ldots\ldots\ldots.(2.3)$$

Again, when we substitute for r_M and T with their value, giving us $2.72\text{x}10^{-3}\ \text{ms}^{-2}$. The nearly perfect agreement between this value and the value Newton obtained using g provides strong evidence of the inverse-square nature of the gravitational force law.

II.1.1 Measuring gravitational constant

The universal gravitational constant G was measured in an important experiment by Henry Cavendish (1731–1810) in 1798. The Cavendish apparatus consists of two small spheres, each of mass m, fixed to the ends of a light horizontal rod suspended by a fine fiber or thin metal wire, as illustrated in **Figure 2.2**. When two large spheres, each of mass M, are placed near the smaller ones, the attractive force between smaller and larger spheres causes the rod to rotate and twist the wire suspension to a new equilibrium orientation. The angle of rotation is measured by the deflection of a light beam reflected from a mirror attached to the vertical suspension. The deflection of the light beam is an effective technique for amplifying the motion. The experiment is carefully repeated with different masses at various separations. In addition to providing a value for G, the results show experimentally that the force is attractive, proportional to the product mM, and inversely proportional to the square of the distance r.

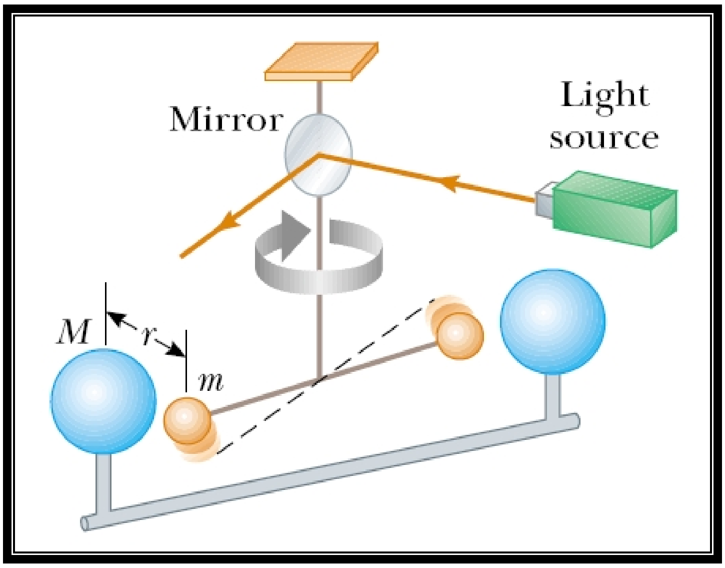

Figure 2.2 Cavendish apparatus for measuring G. The dashed line represents the original position of the rod.

II.1.2 Free-fall acceleration and the gravitational force

When defining mg as the weight of an object of mass m, we refer to g as the magnitude of the free-fall acceleration. Now we are in a position to obtain a more fundamental description of g. Because the magnitude of the force acting on a freely falling object of mass m near the Earth's surface is given by equation (2.1), we can equate mg to this force to obtain

$$\mathbf{mg} = \mathbf{G}\frac{\mathbf{m_E m}}{\mathbf{R_E^2}}$$
$$\mathbf{g} = \mathbf{G}\frac{\mathbf{m_E}}{\mathbf{R_E^2}} \quad \text{.........(2.4)}$$

Now consider an object of mass m located a distance h above the Earth's surface or a distance r from the Earth's centre, where $r = R_E + h$. The magnitude of the gravitational force acting on this object is

$$\mathbf{F_g} = \mathbf{G}\frac{\mathbf{M_E m}}{\mathbf{r^2}} = \mathbf{G}\frac{\mathbf{M_E m}}{(\mathbf{R_E + h})^2} \quad \text{.........(2.5)}$$

The magnitude of the gravitational force acting on the object at this position is also $F_g = mg$, where g is the value of the free-fall acceleration at the altitude h. Substituting this expression for F_g into the last equation shows that g is

$$\mathbf{g} = \mathbf{G}\frac{\mathbf{M_E}}{\mathbf{r^2}} = \mathbf{G}\frac{\mathbf{M_E}}{(\mathbf{R_E + h})^2} \quad \text{.........(2.6)}$$

Thus, it follows that g *decreases* with *increasing altitude.* Because the weight of an object is mg, we see that as $r \to \infty$, its weight approaches zero.

Free-Fall Acceleration *g* at Various Altitudes Above the Earth's Surface

Altitude *h* (km)	*g* (m/s²)
1 000	7.33
2 000	5.68
3 000	4.53
4 000	3.70
5 000	3.08
6 000	2.60
7 000	2.23
8 000	1.93
9 000	1.69
10 000	1.49
50 000	0.13
∞	0

Table 1. The dependant of g on altitude.

Example:

<u>Problem</u> *The International Space Station operates at an altitude of 350 km. When final construction is completed, it will have a weight (measured at the Earth's surface) of 4.22x10⁶ N. What is its weight when in orbit?*

<u>Answer</u> *We first find the mass of the space station from its weight at the surface of the Earth:* $\mathbf{m} = \mathbf{F_g}/\mathbf{g}$ *= 4.31x10⁵ kg*

We use equation (2.6) to find $\mathbf{g} = 8.83\ ms^{-2}$

Using the value of $\mathbf{g}$ *at the location of the station, the station's weight in orbit is:* $\mathbf{F_g} = \mathbf{mg} = 3.80\times10^{6}$ *N. This value is smaller than the value at the Earth's surface which is* 4.22×10^{6} *N.*

II.2 Energy Considerations in Planetary and Satellite Motion

II.2.1 Orbital energy

Consider an object of mass m moving with a speed v in the vicinity of a massive object of mass M, where $M >> m$ (see **Figure 2.3**). The system might be a planet moving around the Sun, a satellite in orbit around the Earth, or a comet making a one-time flyby of the Sun. If we assume that the object of mass M is at rest in an inertial reference frame, then the total mechanical energy E of the two-object system when the objects are separated by a distance r is the sum of the kinetic energy of the object of mass m and the potential energy of the system, given by

$$\mathbf{E = K + U}$$

$$\mathbf{E = \tfrac{1}{2}mv^2 - G\frac{Mm}{r}} \quad \text{……….(2.7)}$$

This equation shows that E may be positive, negative, or zero, depending on the value of v. However, for a bound system, such as the Earth–Sun system, E is necessarily *less than zero* because we have chosen the convention that $U \to 0$ as $r \to \infty$.

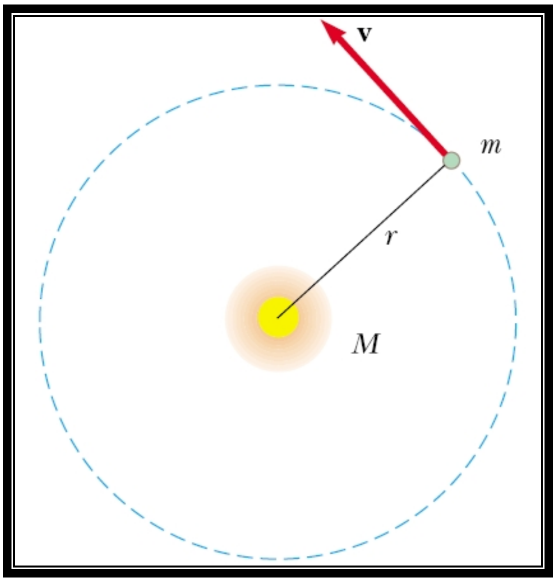

Figure 2.3 An object of mass *m* moving in a circular orbit about a much larger object of mass *M*.

We can easily establish that $E < 0$ for the system consisting of an object of mass m moving in a **circular orbit** about an object of mass $M >> m$. Newton's second law applied to the object of mass m gives

$$G\frac{Mm}{r^2} = ma = m\frac{v^2}{r}$$

Multiplying both sides by r and dividing by 2 and then substituting this into equation (2.7), we obtain

$$E = G\frac{Mm}{2r} - G\frac{Mm}{r}$$
$$E = -G\frac{Mm}{2r} \qquad \text{……….(2.8)}$$

This result clearly shows that the total mechanical energy is negative in the case of circular orbits. Note that the kinetic energy is positive and equal to half the absolute value of the potential energy. The absolute value of E is also equal to the binding energy of the system, because this amount of energy must be provided to the system to move the two objects infinitely far apart. The total mechanical energy is also negative in the case of elliptical orbits. The expression for E for **elliptical orbits** is the same as equation (2.8) with r replaced by the semimajor axis length a:

Example:

<u>Problem</u> *The space shuttle releases a 470-kg communications satellite while in an orbit 280 km above the surface of the Earth. A rocket engine on the satellite boosts it into a geosynchronous orbit (4.23x10*^{4} *km from the centre of the Earth), which is an orbit in which the satellite stays directly over a single location on the Earth. How much energy does the engine have to provide?*

<u>Answer</u> *We first determine the initial orbital radius (not the altitude above the Earth's surface):* $\boldsymbol{r_A} = \boldsymbol{R_E} + h = 6.65\text{x}10^5$ *m*

The energy required from the engine to boost the satellite from its initial to final position is calculated using equation (2.11). The result is: ***1.19x10***^{10} ***J*** *(actually, we must account for the changing mass of the spacecraft as it ejects burned fuel, something we have not done here).*

$$\mathbf{E = -G\frac{Mm}{2a}} \quad \text{……….(2.9)}$$

Furthermore, the total energy is constant if we assume that the system is isolated. Therefore, as the object of mass *m* moves from, say, A to B, the total energy remains constant and equation (2.7) gives

$$\mathbf{E = \tfrac{1}{2}mv_A^2 - G\frac{Mm}{r_A} = \tfrac{1}{2}mv_B^2 - G\frac{Mm}{r_B}} \quad \text{……….(2.10)}$$

or

$$\begin{aligned} &\mathbf{E = -G\frac{Mm}{r_A} = -G\frac{Mm}{r_B}} \\ &\mathbf{\Delta E = -G\frac{Mm}{2}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)} \end{aligned} \quad \text{……….(2.11)}$$

II.2.2 Escape speed

Suppose an object of mass *m* is projected vertically upward from the Earth's surface with an initial speed v_i , as illustrated in **Figure 2.4**. We can use energy considerations to find the minimum value of the initial speed needed to allow the object to move infinitely far away from the Earth. Equation (2.7) gives the total energy of the system at any point. At the surface of the Earth, $v = v_i$ and r

$= r_i = R_E$. When the object reaches its maximum altitude, $v = v_f = 0$ and $r = r_f = r_{\max}$. Because the total energy of the system is constant, substituting these conditions into equation (2.7) gives

$$\tfrac{1}{2}mv_i^2 - G\frac{M_E m}{R_E} = -G\frac{M_E m}{r_{max}}$$

Solving for v_i^2 gives

$$v_i^2 = 2GM_E m\left(\frac{1}{R_E} - \frac{1}{r_{max}}\right) \quad \text{……….(2.12)}$$

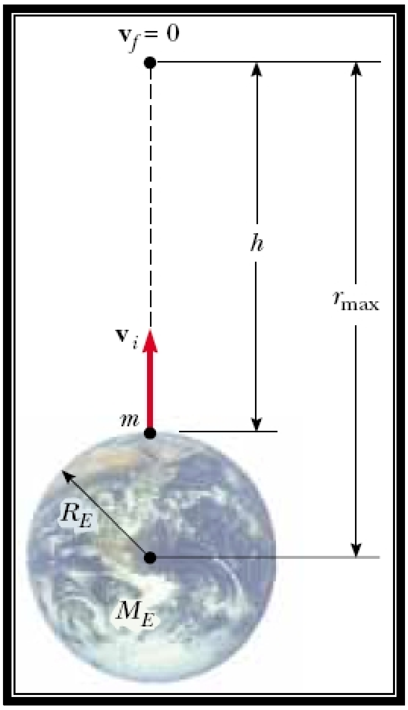

Figure 2.4 An object of mass m projected upward from the Earth's surface with an initial speed v_i reaches a maximum altitude h.

We are now in a position to calculate escape speed, which is the minimum speed the object must have at the Earth's surface in order to approach an infinite separation distance from the Earth. Traveling at this minimum speed, the object continues to move farther and farther away from the Earth as its speed asymptotically approaches zero. Letting $r_{\max} \to \infty$ in equation (2.12) and taking $v_i = v_{esc}$, we obtain

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}} \quad \text{……….(2.13)}$$

Note that this expression for v_{esc} is independent of the mass of the object. In other words, a spacecraft has the same escape speed as a molecule. Furthermore, the result is independent of the direction of the velocity and ignores air resistance.

If the object is given an initial speed equal to v_{esc}, the total energy of the system is equal to zero. This can be seen by noting that when $r \rightarrow \infty$, the object's kinetic energy and the potential energy of the system are both zero. If v_i is greater than v_{esc}, the total energy of the system is greater than zero and the object has some residual kinetic energy as $r \rightarrow \infty$.

Example:

Problem *Calculate the escape speed from the Earth for a 5,000-kg spacecraft, and determine the kinetic energy it must have at the Earth's surface in order to move infinitely far away from the Earth.*

Answer *We use equation (2.13) to find the escape speed:*

$$\mathbf{v_{esc} = \sqrt{\frac{2GM_E}{R_E}} = 1.12 \times 10^4\,ms^{-1}}$$

The kinetic energy of the spacecraft is

$$\mathbf{K = \tfrac{1}{2}mv_{esc}^2 = 3.14 \times 10^{11}\,J}$$

II.3 Tidal Force

A number of phenomena on Earth depend on the fact that the gravitational force exerted by the Moon (or the Sun) on the Earth is slightly different at different parts of the Earth. Any effect which depends on the *difference* between the gravitational forces on opposite sides on an object is called a *tidal effect*.

If we have an object of mass m with radius r, a distance d from an object of mass M, the gravitational attraction on the object of mass m at the point on the closest to M is

$$F_g = G\frac{Mm}{(d-r)^2}$$

while the point farthest from *M* feels a gravitational attraction of

$$F_g = G\frac{Mm}{(d+r)^2}$$

The near point is being pulled *more strongly* toward *M*, while the far point is being pulled *less strongly*, than the centre. Relative to the centre, the near point is pulled *toward M*, while the far point is pushed *away* from *M*. The difference between these forces relative to the centre is the tidal force, those are

$$F_{tides} = (+/-)\frac{2GMmr}{d^3} \quad \text{..........(2.14)}$$

Tidal forces are responsible for many effects in our solar system, from changing sea levels to geological activity on, and the ultimate fate of, distant moons.

Now let's consider the Earth–Moon system. If Earth were rotating at the same rate as the Moon's orbital period, the bulge would point straight at the Moon as illustrated in **Figure 2.5** (left). However, the Earth is rotating considerably faster than the Moon's orbital period, and the bulge, which can't move as quickly, is pushed ahead, and points ahead of the Moon's position (**Figure 2.5** (centre)). The Moon's gravity "pulls back" on the advanced bulge (**Figure 2.5** (right)).

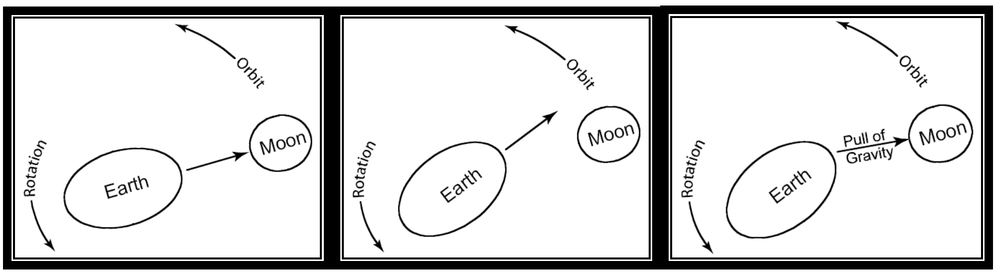

Figure 2.5 Tidal force on Earth–Moon system.

This backward torque gradually slows the rotation of the earth. To conserve the angular momentum of the system, the radius of the Moon's orbit gradually increases. If the Earth were rotating more *slowly* than the Moon's orbit, the bulge would lag *behind* the connecting line, and the Moon would be gradually speeding up the Earth’s rotation, and moving closer to conserve angular momentum.

II.3.1 The effects of tidal force

II.3.1.1 tides on earth

The tidal bulge, when it is over an ocean, presents itself as a rise in sea level. A 90° later in revolution, when the bulge is at right angle to the ocean, a reduced sea level is experienced.

The Moon is not alone in pulling on the Earth. The distant Sun is so massive that it also exerts a significant tidal force, about half the strength of the Moon’s. When the Earth, Moon, and Sun are in a straight line (Full Moon and New Moon), the Sun’s effect is added to the Moon’s, creating the highest tides, called Spring Tides. At Quadrature, when the Sun and Moon form a 90° angle with Earth, the Sun’s effect is minimized, and the lowest tides, Neap Tides, occur.

II.3.1.2 resonances

Tidal forces cause the rotations and orbits of bodies to be synchronized in integer ratios. The most evident case is *Synchronous Rotation*, where a satellite rotates at the same rate as its orbital period, keeping the same face toward its parent. All large satellites in our solar system, including our Moon, do this. Above, we discussed the slowing of Earth’s orbit by the Moon pulling on the advanced tidal bulge. In the distant past, when the Moon was rotating at some other speed, Earth would have had the same effect on the Moon, slowing or speeding its rotation until it reached a rotational speed where the bulge precisely faced the Earth.

Higher-Order Resonances also occur in the solar system. For example, the orbits of Io, Europa, and Ganymede are in a 1:2:4 resonance. Relationships such as these result from interaction of the bodies' gravitational forces on one another.

II.3.1.3 tidal heating

Tidal forces can also internally heat a satellite. Satellites rotate at a constant speed but, in an eccentric orbit, the orbital speed varies with the distance from the primary (Kepler's 2nd law). Thus the tidal bulge of a tidally locked satellite cannot always point precisely at its primary. At some points in the orbit, the bulge will be pointed ahead of or behind where it should be, and the primary will be exerting a small pull on the bulge. The friction of the bulge being pulled back and forth through the solid body of the satellite heats the interior of the body.

Io is an extreme example of tidal heating. While Io's orbit is nearly circular, it frequently encounters Europa and Ganymede, causing temporary changes in the tidal forces it feels, moving the tidal bulge around the planet. In the Jupiter's intense gravity the friction is significant, and they predicted Io should be hot enough for Volcanism. Unmanned spacecraft, Voyager-1, photographed active volcanoes on Io, and we now know it to be the most volcanically active object in the solar system, with internal temperatures of 2,000 K. Tidal heating is also probably responsible for keeping the interior of Europa warm enough for the liquid water that is suspected to exist below the ice surface, and tidal resonance with Saturn's Dione is thought to power volcanism on Enceladus.

II.3.1.4 the roche limit

In 1848, Astronomer Edouard Roche noted that, if a satellite was held together mainly by its own gravitational attraction, there would be a minimum distance from the primary inside which the tidal forces of the primary would exceed the satellite's binding forces and would tear it apart.

The Roche Limit for two bodies is approximated by a function of their densities, ρ:

$$\mathbf{R_L = 2.456R\left(\frac{\rho_{planet}}{\rho_{satellite}}\right)^{\frac{1}{3}}} \quad \text{..........(2.15)}$$

where R is the planet's radius, and the ρ values are the densities. For typical satellites, a common approximation is that the Roche Limit is 2.5 times the radius of the primary planet.

The Roche Limit applies only to fluid bodies held together entirely by gravitation. Small satellites and moons can survive inside their primary's Roche Limit because their electrochemical bonds are more significant than their gravitational bonds. For small rocky satellites, the Roche Limit is approximated as 1.44 primary radii.

II.3.1.5 rings and satellites

All large satellites in the solar system orbit outside their planet's Roche Limit. Small rocky satellites (usually under 100 km in diameter) can exist inside the Roche Limit, and there are examples of this with satellites of Jupiter, Uranus, and Neptune. All the Jovian planets are now known to have rings, all *inside* their primary's Roche Limit, when the Roche calculations for a relatively diffuse body are used.

Saturn's rings composed of small particles, and Roche suggested they were a former satellite that broke up because it was inside the Roche Limit. However, tidal forces would have prevented a satellite from forming in the first place, so a mechanism to form a satellite elsewhere and then move it inside the limit was needed. It is possible that a separate object was captured by Saturn's gravity and pulled inside the Roche Limit, where it was destroyed (as happened with comet Shoemaker-Levy 9 at Jupiter), or that a moon, originally further out, spiralled into the planet through tidal interactions (as is happening with Phobos and Triton).

An alternate theory is that the rings consist of original material from the solar nebula, prevented by tidal forces from ever becoming a satellite. In this case, a mechanism is needed to explain the sharp boundaries of the ring systems, and why they have not become diffuse with time. Shepherding effects of small nearby satellites are the proposed solution to the sharp boundaries, and replenishment of the rings with dust stripped off satellites could keep them from fading with time. Photographs of unmanned spacecraft module, Galileo, from 1996–1997 show dust being pulled off Amalthea and Thebe and into the rings of Jupiter, supporting this theory.

II.3.1.6 catastrophic events

Finally, occasional extraordinary events serve to demonstrate the power of tidal forces. Neptune's moon Triton is due for such an event. Like all retrograde satellites, its orbit is decaying, and it will fall below Neptune's Roche limit and be destroyed in 100 million to 1 billion years.

More recently, when comet Shoemaker-Levy 9 was discovered in 1993, it had already broken up into more than twenty pieces when it passed within 21,000 km of Jupiter (Jupiter's Roche limit is 175,000 km, so the tidal stresses at 21,000 would have been enormous.) The pieces spiralled in to impact the planet in July 1994. This event showed that theories of stray objects being destroyed and contributing to planetary evolution are credible.

II.4 Kepler's Law and the Motion of Planets

People have observed the movements of the planets, stars, and other celestial objects for thousands of years. In early history, scientists regarded the Earth as the center of the Universe. This so-called *geocentric model* was elaborated and formalized by the Greek astronomer Claudius Ptolemy (c. 100–c. 170) in the second century A.D. and was accepted for the next 1,400 years. In 1543 the Polish astronomer Nicolaus Copernicus (1473–1543) suggested that the Earth and the other planets revolved in circular orbits around the Sun (the *heliocentric model*).

The Danish astronomer Tycho Brahe (1546–1601) wanted to determine how the heavens were constructed, and thus he developed a program to determine the positions of both stars and planets. It is interesting to note that those observations of the planets and 777 stars visible to the naked eye were carried out with only a large sextant and a compass (the telescope had not yet been invented).

The German astronomer Johannes Kepler was Brahe's assistant for a short while before Brahe's death, whereupon he acquired his mentor's astronomical data and spent 16 years trying to deduce a mathematical model for the motion of the planets. Such data are difficult to sort out because the Earth is also in motion around the Sun. After many laborious calculations, Kepler found that Brahe's data on the revolution of Mars around the Sun provided the answer. Kepler's complete analysis of planetary motion is summarized in three statements known as **Kepler's laws**:

1. *All planets move in **elliptical orbits** with the Sun at one focus.*
2. *The radius vector drawn from the Sun to a planet sweeps out **equal areas in equal time intervals**.*
3. *The **square of the orbital period** of any planet is **proportional to** the **cube of the semimajor axis** of the elliptical orbit.*

II.4.1 Kepler's first law

Figure 2.6 shows the geometry of an ellipse, which serves as our model for the elliptical orbit of a planet. An ellipse is mathematically defined by choosing two points F_1 and F_2, each of which is a called a focus, and then drawing a curve through points for which the sum of the distances r_1 and r_2 from F_1 and F_2, respectively, is a constant. The longest distance through the center between points on the ellipse (and passing through both foci) is called the major axis, and this distance is $2a$. In **Figure 2.6**, the major axis is drawn along the x direction. The distance a is called the semimajor axis. Similarly, the shortest distance through the center between points on the ellipse is called the minor axis of length $2b$, where the distance b is the semiminor axis. Either focus of the ellipse is located at a distance c from the center of the ellipse, where $a^2 = b^2 + c^2$. In the elliptical orbit of a planet around the Sun, the Sun is at one focus of the ellipse. There is nothing at the other focus.

The eccentricity of an ellipse is defined as $e = c/a$ and describes the general shape of the ellipse. For a circle, $c = 0$, and the eccentricity is therefore zero. The smaller b is than a, the shorter the ellipse is along the y direction compared to its extent in the x direction in **Figure 2.6**. As b decreases, c increases, and the eccentricity e increases.

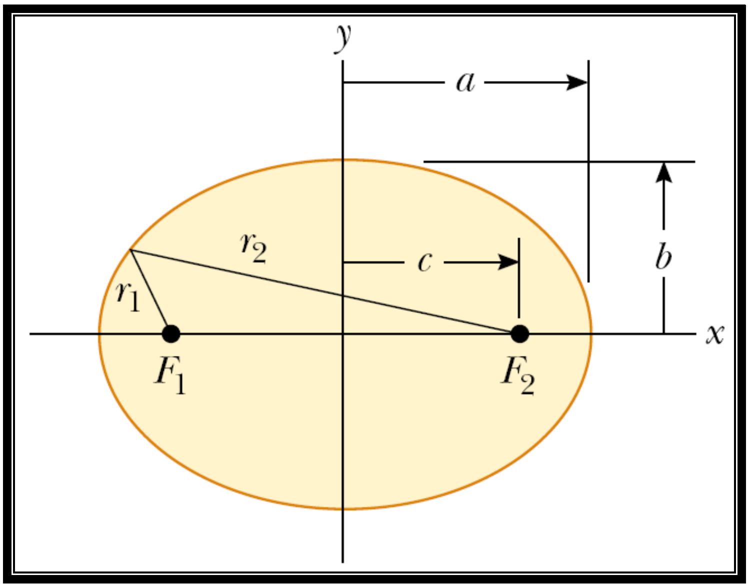

Figure 2.6 Plot of an ellipse. The semimajor axis has length a, and the semiminor axis has length b. Each focus is located at a distance c from the center on each side of the center.

Now imagine a planet in an elliptical orbit such as that shown in **Figure 2.6**, with the Sun at focus F_2. When the planet is at the far left in the diagram, the distance between the planet and the Sun is $a + c$. This point is called the *aphelion*, where the planet is the farthest away from the Sun that it can be in the orbit. (For an object in orbit around the Earth, this point is called the *apogee*). Conversely, when the planet is at the right end of the ellipse, the point is called the *perihelion* (for an Earth orbit, the *perigee*), and the distance between the planet and the Sun is $a - c$.

Kepler's first law is a direct result of the inverse square nature of the gravitational force. Circular and elliptical orbits are the allowed shapes of orbits for objects that are *bound* to the gravitational force center. These objects include planets, asteroids, and comets that move repeatedly around the Sun, as well as moons orbiting a planet. There could also be *unbound* objects, such as a meteoroid from deep space that might pass by the Sun once and then never return. The gravitational force between the Sun and these objects also varies as the inverse square of the separation distance, and the allowed paths for these objects include parabolas ($e = 1$) and hyperbolas ($e > 1$).

II.4.2 Kepler's second law

Kepler's second law can be shown to be a consequence of angular momentum conservation as follows. Consider a planet of mass M_P moving about the Sun in an elliptical orbit (see **Figure 2.7a**). Let us consider the planet as a system. We will model the Sun to be so much more massive than the planet that the Sun does not move. The gravitational force acting on the planet is a central force, always along the radius vector, directed toward the Sun. The torque on the planet due to this central force is clearly zero, because **F** is parallel to **r**. That is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{0}$$

Recall that the external net torque on a system equals the time rate of change of angular momentum of the system; that is, $\tau = d\mathbf{L}/dt$. Therefore, because $\tau = 0$, **the angular momentum L of the planet is a constant of the motion**:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{M_p r} \times \mathbf{v} = \text{constant}$$

We can relate this result to the following geometric consideration. In a time interval dt, the radius vector **r** in **Figure 2.7b** sweeps out the area dA, which equals half the area $|\mathbf{r} \times \mathbf{dr}|$ of the parallelogram formed by the vectors **r** and $d\mathbf{r}$. Because the displacement of the planet in the time interval dt is given by $d\mathbf{r} = \mathbf{v}dt$, we have

$$\mathbf{dA} = \tfrac{1}{2}|\mathbf{r} \times \mathbf{dr}| = \tfrac{1}{2}|\mathbf{r} \times \mathbf{v}\mathbf{dt}| = \frac{\mathbf{L}}{\mathbf{2M_P}}\mathbf{dt}$$

$$\frac{\mathbf{dA}}{\mathbf{dt}} = \frac{\mathbf{L}}{\mathbf{2M_P}} = \text{constant} \quad \ldots\ldots\ldots(2.16)$$

where L and M_P are both constants. Thus, we conclude that the radius vector from the Sun to any planet sweeps out equal areas in equal times.

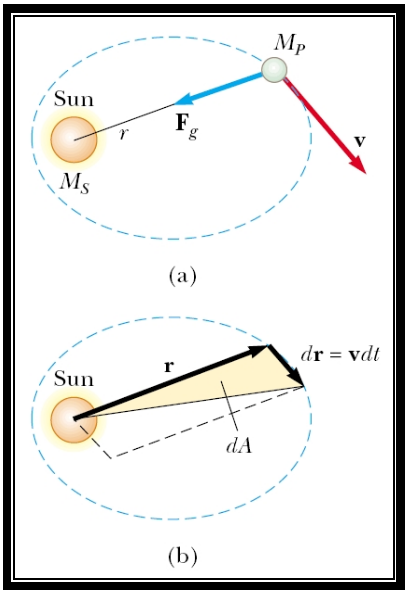

Figure 2.7 (a) The gravitational force acting on a planet is directed toward the Sun. (b) As a planet orbits the Sun, the area swept out by the radius vector in a time interval dt is equal to half the area of the parallelogram formed by the vectors $\mathbf{r}$ and $d\mathbf{r} = \mathbf{v}dt$.

II.4.3 Kepler's third law

It is informative to show that Kepler's third law can be predicted from the inversesquare law for circular orbits. Consider a planet of mass M_P that is assumed to be moving about the Sun (mass M_S) in a circular orbit, as in **Figure 2.8**. Because the gravitational force provides the centripetal acceleration of the planet as it moves in a circle, we use Newton's second law for a particle in uniform circular motion.

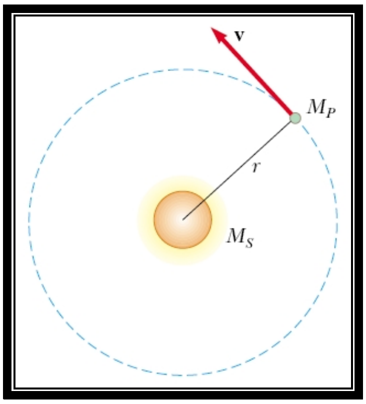

Figure 2.8 A planet of mass M_P moving in a circular orbit around the Sun. The orbits of all planets except Mercury are nearly circular.

We process as follows,

$$G\frac{M_S M_P}{r^2} = \frac{M_P v^2}{r}$$

$$G\frac{M_S M_P}{r^2} = \frac{M_P\left(\frac{2\pi r}{T}\right)^2}{r}$$

$$T^2 = \left(\frac{4\pi^2}{GM_S}\right)r^3 = (\text{Constant})r^3$$

The last line in the above process is also valid for elliptical orbits if we replace *r* with the length *a* of the semimajor axis,

$$T^2 = \left(\frac{4\pi^2}{GM_S}\right)a^3 = (\text{constant})a^3 \text{ ....(2.17)}$$

Equation (2.17) is Kepler's third law. Because the semimajor axis of a circular orbit is its radius, equation (2.17) is valid for both circular and elliptical orbits. Note that the constant of proportionality K_S is independent of the mass of the planet. Equation (2.17) is therefore valid for *any* planet. If we were to consider the orbit of a satellite such as the Moon about the Earth, then the constant would have a different value, with the Sun's mass replaced by the Earth's mass.

Example:

Problem *Calculate the mass of the Sun using the fact that the period of the Earth's orbit around the Sun is* $3.156\text{x}10^7$ *s and its distance from the Sun is* $1.496\text{x}10^{11}$ *m.*

Answer *We use equation (2.17) to find the mass of the Sun:*

$M_S = (4\pi^2 r^3)/(GT^2) = 1.99\text{x}10^{30}$ kg

Astronomy Laboratory

THE MASS OF JUPITER

Procedure:

You are going to use software from *CLEA Project: The revolution of the moons of jupiter* to do this laboratory activity.

1. ***Start-up***
   Double-click on the icon Jupiter's Moons.
   After the program is activated it will request a Start Date and Time for your observing session. Your Start Date and Time can be obtained from your TA.

2. ***The Telescope Field of View and Readouts***
   After you have entered this information into the computer, it will point the telescope at Jupiter and provide a display similar to that shown below. Jupiter is displayed in the center of the screen. To either side are the Galilean moons. Even at high magnifications they appear only as points of light with no visible surface.

   The date and the Universal Time are displayed in the lower left corner. There is also a number labeled JD, which stands for Julian Day. This is the number of days (and fractions of days) since noon on Jan 1, 4713 BC, and is the standard system astronomers use to record and communicate dates.

   There are four buttons on the screen marked 100x, 200x, 300x and 400x. The magnification of the telescope can be controlled by clicking these buttons. Try clicking on them to see how it changes the view. The current telescope magnification is shown at the upper left corner.

   The "Next" button steps ahead one time interval. To end your observing session, select QUIT from the File menu. You cannot continue where you left off if you QUIT the program.

3. ***Position Measurement***
   In order to measure a moon's position, move the cursor into the telescope field of view, then press and hold the mouse button. The cursor becomes a cross and the measurement software is activated. Carefully center the cross on a moon and read the value next to the lower 'X' in the lower righthand corner. The number is the distance of the cursor from the center of Jupiter, measured in JuD (to make sure that you are clear on this, check that the edge of Jupiter is 0.5 JuD). The direction of the moon is given by an E or W for east and west.

Lab Skills and Objectives

- **Be able to determine the mass of Jupiter by measuring the orbital properties of Jupiter's moons and analyzing their motions using Kepler's third law**

Sometimes a moon will be "missing". In this case, first go to the lowest magnification and largest field of view. If you don't see four moons, then go to the highest power to see if one is in front of Jupiter (this is called a transit). If the missing moon is not in front of Jupiter, it must be behind the planet. In this case, record its distance as zero. Of course, this measurement could be in error by as much as 0.5 JuD. Once you have measured the positions of all four moons, click "Next" to advance in time by one time step.

4. ***Recording Data***

In this lab your data are your position measurements for the moons. On this page is a table for recording your data. Record your observations as follows:

Column 1: Write down the Julian Day. Since you likely will not be observing for more than a year in time, you can just record the last few digits. For example you could put down 9883.43 instead of 2449883.43 to save some writing.

Columns 2 through 5: Record each moon's position, in the order that you see them from east to west (left to right). Remember to specify if they are east (left) or west (right) of Jupiter. For example, you could write 2.21W to mean one moon was 2.21 JuD west (right) of Jupiter's center.

Data Table

(1) Julian Day	(2)	(3)	(4)	(5)
2449883.00	9.81E	3.75E	1.72W	2.11W

5. ***Data Presentation – Graphical***

More often than not, the graphical display of data makes trends and correlations readily apparent. Often the basis of a discovery is simply recognizing what to plot against what! Given the data in this lab the proper plot is evidently the position of each moon versus time, specifically Julian Date. The number of days since the first observation is along the x-axis and distance from Jupiter is along the y-axis.

6. ***Identifying Moons, and Determining Orbital Periods and Semi-Major Axes***
Before you can determine the periods and semi-major axes of the moons' orbits, you need to figure out which measurement goes with which moon at any given time. Since nature doesn't label the moons for us, this can only be done by looking for patterns in the data. Fortunately, the orbits of the Galilean moons are circular so that the path of each moon in your graph of position versus time will look like a sine curve. For any one moon the sine curve will be symmetric, with all maxima and minima the same distance from 0.

Now you can derive the essential quantities -period P and semi-major axis a -from your observations, and from them derive the mass of Jupiter. For each moon you will have a curve something like the sine curve. The semi-major axis of the orbit is given by the maximum distance from Jupiter at which you observed the moon. Equivalently, the semi-major axis is given by the maximum or minimum value of your sine curve. Of course, since your measures are in JuD, so will be your value for the semi-major axis of the orbit.

Finally, your measures are in days and JuD, while Kepler's Third Law requires years and AU. To convert your period in days to a period in years, simply divide by 365.25 days in a year. To convert your semi-major axis in JuD to a semi-major axis in AU, divide by 1050 Jupiter diameters in an AU.

7. ***Determining the Mass of Jupiter – Four Times!***
Use the Kepler's third law to obtain the mass of Jupiter from its four innermost satellites. Compare your result with the scientifically accepted value.