

APPLICATION OF PHYSICS TO FINANCE AND ECONOMICS: QUANTUM FIELD THEORY IN FORWARD RATES PARAMETERS

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Abstract. Quantum theory in physics is used to model secondary financial markets. Contrary to stochastic description, the formalism emphasizes the importance of trading in determining the value of security. All possible realization of investors holding securities and cash is as the basis of the Hilbert space of market states. The asymptotic volatility of a stock related to long-term probability that is traded. This paper investigates volatility of forward rates in secondary financial market. In recent formulation of a quantum field theory of forward rates, the volatility of the forward rates was taken to be deterministic. The field theory of the forward rates is generalized to the case of stochastic volatility. Two cases are analyzed, firstly when volatility is taken to be a function of forward rates, and secondly when volatility is taken to be an independent quantum field. Since volatility is a positive quantum field, the full theory turns out to be an interacting non-linear quantum field in two dimensions. The state space and Hamiltonian for the interacting theory are obtained, and shown to have a nontrivial structure due to the manifold move with constant velocity.

Keywords: volatility, forward rates, quantum field theory, arbitrage

1. Introduction

There are 7 financial instruments i.e: stock, bonds, convertible bonds, rights, warrant, term structures, and asset back securities. Furthermore, there are also derivative financial instruments i.e: options (both forward or futures-based) and swap. This paper focus on forward foreign exchange (forex). Forex forward contract is a contract between a bank with its partner (it could be a bank) make a deal to deliver at the prescribed time in the future, a prescribed amount of money, and the value of the money are fixed at the sign of the contract.

If the value of the contract is larger, the possibility of different exercise prices (higher or lower) will larger also, which define as volatility. Volatility is regarded as referenced as deviation standard the fluctuation financial instrument as a time series. It is essential to the debt market with have wide ranging application in finance. The most popular used model for the forward rates is the Heath-Jarrow-Morton (HJM) [1], and there are number of ways HJM model can be generalized. In [2] and [3] was introduced the correlation between forward rates and its various maturity and in [4], [5] forward rates was modeled as a stochastic string.

The application of technical of physics in finance [6] [7] have proved useful in the application, especially the using of path integral in various of finance problems [8]. In [9], path integral techniques has been applied to investigate a security product with stochastic volatility. In [10], HJM model is generalized with assumption forward rates as a quantum field. Empirical study by [11] showed that quantum field theory for the forward rates when applied with market data have a good result on prediction.

The volatility of the forward rates is an essential measurement to determine the degree of forward fluctuation. In the model studied [10] volatility is taken as deterministic variable of forward function. The question naturally arises as whether the volatility itself should be considered to be randomly fluctuating quantities. The volatility of the volatility is an accurate measure of degree to which volatility is random. Market data for the Eurodollar futures provides a fairly accurate estimate of the forward rates for the US dollar, and also yields its volatility of volatility of the forward rates.

The fluctuation of the volatility of eurodollar in [17] are about 10% of the forward rates, and hence significant. We conclude from the data that the volatility of the forward rates needs to be treated as a fluctuating quantum (stochastic) field. The widely studied HJM model [1] has been further developed by [12] to account for stochastic volatility. Amin dan Ng [13] studied the market data of Eurodollar option to obtain the implied of the forward rates volatility, and Bouchaud et. Al [14] analyzed the future contracts for the forward rates. Both reference concluded that many features of the market, and in particular the stochastic volatility of forward curve, could not be fully explained in the HJM model – framework.

The model for the forward rates proposed in [10] is a field theoretic generalization of the HJM model, and so it is natural to extend the field theory model to account for stochastic volatility of the forward rates. In contrast to quantum field theory, the formulation of the forward rates as a stochastic string in [4], [5] cannot be extended to the case where volatility is stochastic due to nonlinearities inherent in the problem.

2. Quantum Field Theory for Stochastic Volatility of the Forward Rates

The forward rates are the collection of interest rates for a contract entered into at time t for an overnight loan at time $x > t$. At any instant t , there exists in the market forward rates for a duration of T_{FR} in the future; for example, if t refers to present time t_0 , then one has forward rates from t_0 till time $t_0 + T_{FR}$ in the future. In the market, T_{FR} is about 30 years, and hence we have $T_{FR} > 30$ years. In general, at any time t , all the forward rates exist till time $t + T_{FR}$ [10]. The forward rates at time t are denoted by $f(t, x)$, with $t < x < t + T_{FR}$, and constitute the forward rate curve. Since at any instant t there are infinitely many forward rates, it resembles a (non-relativistic) quantum string. Hence we need an infinite number of independent variables to describe its random evolution. The generic quantity describing such a system is a quantum field [15]. For modeling the forward rates and Treasury Bonds, we consequently need to study a two-dimensional quantum field on a finite Euclidean domain. We consider the forward rates $f(t, x)$ to be a quantum field; that is, $f(t, x)$ is taken to be an independent random variable for each x and each t . For notational simplicity we consider both t and x continuous, and discredited these parameters only when we need to discuss the time evolution of the system in some detail.

In Section 2, we briefly review the quantum field theoretic formulation given in [10] of the forward rates with deterministic volatility. In Section 3 the case of stochastic volatility is analyzed, and which can be done in two different ways. Firstly, volatility can be considered to be a function of the (stochastic) forward rates, and secondly volatility can be considered to be an independent quantum field. Both these cases are analyzed. The resulting theories are seen to be highly non-trivial non-linear quantum field theories.

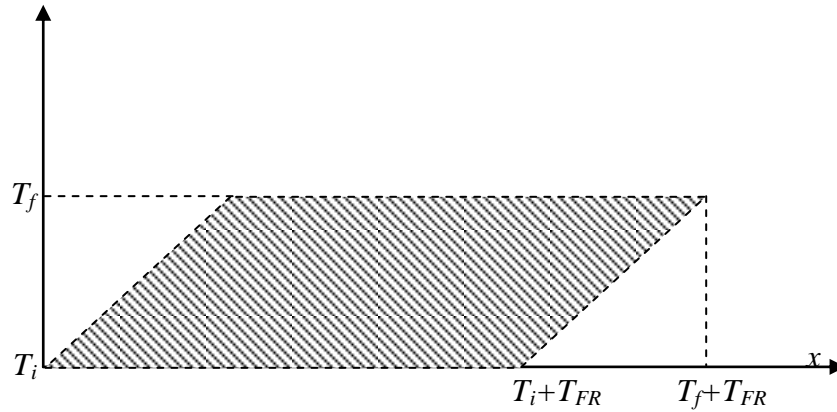
In Section 4 the underlying state space and operators of the forward rates quantum field is defined. In particular the generator of infinitesimal time evolution of the forward rates, namely the Hamiltonian, is derived for the two cases of stochastic volatility. In Section 5 the Hamiltonian for the forward rates with stochastic volatility is derived. In Section 6 a Hamiltonian formulation of the condition of no arbitrage is derived. In Section 7 the no arbitrage constraint for the case of stochastic volatility is solved exactly using the Hamiltonian formulation. And lastly, in Section 8 the results obtained are discussed, and some remaining issues are addressed.

2.1 The Lagrangian for Forward Rates with Deterministic Volatility

We first briefly recapitulate the salient features of the field theory of the forward rates with deterministic volatility [10]. For the sake of concreteness, consider the forward rates starting from some initial time T_i to a future time $t = T_f$. Since all the forward rates $f(t, x)$ are always for the future, we have $x > t$; hence the quantum field $f(t, x)$ is defined on the domain in the shape of a parallelogram that \mathcal{P} is bounded by parallel lines $x = t$ and $x = T_{FR} + t$ in the maturity direction, and by the lines $t = T_i$ and $t = T_f$ in the time direction, as shown in Figure 3.1.

Every point inside the domain variable $f(t, x)$ represents an independent integration. The field theory interpretation of the evolution of the forward rates, as expressed in the domain \mathcal{P} , is that of a (non-relativistic) quantum string moving with unit velocity in the x (maturity) direction.

Figure 3.1. Domain \mathcal{P} of Forward Rates



Since we know from the HJM-model that the forward rates have a drift velocity $\alpha(t, x)$ and volatility $\sigma(t, x)$, these have to appear directly in the Lagrangian for the forward rates. To define a Lagrangian, we firstly need a kinetic term, $\mathcal{L}_{\text{kinetic}}$, that is necessary to have a standard time evolution for the forward rates. We need to introduce another term to constrain the change of shape of forward rates in the maturity direction. The analogy of this in the case of an ordinary string is a potential term in the Lagrangian which attenuates sharp changes in the shape of the string, since the shape of the string stores potential energy. To model a similar property for the forward rates we cannot use a simple tension-like term $\left(\frac{\partial f}{\partial x}\right)^2$ in the Lagrangian since, as we will show in Section 7, this term is ruled out by the condition of no arbitrage. The no arbitrage condition requires that the forward rates Lagrangian contain higher order derivative terms, essentially a term of the form $\left(\frac{\partial^2 f}{\partial x \partial t}\right)^2$ such string systems have been studied in [16] and are said to be strings with finite rigidity. Such a term yields a term in the forward rates Lagrangian, namely $\mathcal{L}_{\text{rigidity}}$, with a new parameter μ ; the rigidity of the forward rates is then given by $\frac{1}{\mu^2}$ and quantifies the strength of the fluctuations of the forward rates in the time-to-maturity direction x . In the limit of $\mu \rightarrow 0$ we recover (up to some rescalings) the HJM-model, and which correspond to an infinitely rigid string.

The action for the forward rates is given by :

$$S[f] = \int_{T_i}^{T_f} dt \int_t^{t+T_{FR}} dx \mathcal{L}[f] \tag{1}$$

$$\equiv \int_{\mathcal{P}} \mathcal{L}[f] \tag{2}$$

With Lagrangian density $\mathcal{L}[f]$ given by:

$$\mathcal{L}[f] = \mathcal{L}_{\text{kinetic}}[f] + \mathcal{L}_{\text{rigidity}}[f] \tag{3}$$

$$= -\frac{1}{2} \left[\left\{ \frac{\frac{\partial f(t,x)}{\partial t} - \alpha(t,x)}{\sigma(t,x)} \right\}^2 + \frac{1}{\mu^2} \left\{ \frac{\partial}{\partial x} \left(\frac{\frac{\partial f(t,x)}{\partial t} - \alpha(t,x)}{\sigma(t,x)} \right) \right\}^2 \right]$$

$$-\infty \leq f(t,x) \leq +\infty \tag{4}$$

The presence of the second term in the action given in eq.(3) is not ruled out by no arbitrage [14], and an empirical study [11] provides strong evidence for this term in the evolution of the forward rates. In summary, we see that the forward rates behave like a quantum string, with a time and space dependent drift velocity $\alpha(t, x)$, an effective mass given by $\sigma(t, x)$, and string rigidity proportional to $\frac{1}{\sigma(t,x)}$ and $\frac{1}{\mu^2}$.

Since the field theory is defined on a finite domain \mathbf{P} as shown in Figure 3.1, we need to specify the boundary conditions on all the four boundaries of the finite parallelogram.

a. Fixed (Dirichlet) Initial and Final Conditions

Fixed (Dirichlet) Initial and Final Conditions in t direction given by:

$$T_i(T_f) < x < T_i(T_f) + T_{FR} : f(T_i, x), f(T_f, x) \tag{5}$$

Specified initial and final forward rate curves.

b. Free (Neumann) Boundary Conditions

To specify the boundary condition in the maturity direction, one needs to analyze the action given in eq.(1) and impose the condition that there be no surface terms in the action. A straightforward analysis yields the following version of the Neumann condition:

$$T_i < t < T_f, \frac{\partial}{\partial x} \left(\frac{\frac{\partial f(t,x)}{\partial t} - \alpha(t,x)}{\sigma(t,x)} \right) = 0 \tag{6}$$

and

$$x = t \text{ atau } x = t + T_{FR} \tag{7}$$

The quantum field theory of the forward rates is defined by the Feynman path integral by integrating over all configurations, and yields of $f(t, x)$ and :

$$Z = \int Df e^{S[f]} \tag{8}$$

$$\int Df = \prod_{(t,x) \in \mathbf{P}} \int_{-\infty}^{+\infty} df(t,x) \tag{9}$$

Note that $e^{S[f]}/Z$ is the probability for different field configurations to occur when the functional integral over $f(t, x)$ is performed.

3. Lagrangian for Forward Rates with Stochastic Volatility

To render the volatility function $\sigma(t, x)$ stochastic, in the formalism of quantum field theory, requires that we elevate $\sigma(t, x)$ from a deterministic function into random function, namely into a quantum field. There are essentially two ways of elevating volatility to a stochastic quantity, namely to either.

- a. Consider it be a function of the forward rate $f(t, x)$, or else ;
- b. To consider it to be another independent quantum field $\sigma(t, x)$.

We study both these possibilities.

3.1. Volatility a function of the Forward Rates

We consider the first case where volatility is rendered stochastic by making it a function of the forward rates [13]. The standard models using this approach consider that volatility is given by :

$$\sigma(t, x), f(t, x) = \sigma_0(t, x) f^v(t, x) \tag{10}$$

with

$$\sigma_0(t, x) : \text{deterministic function} \tag{11}$$

Since volatility $\sigma(t, x) > 0$, we must have $f(t, x) > 0$, Hence, in contrast to eq. (4), we have:

$$f(t, x) = f_0 e^{\phi(t,x)} > 0 ; -\infty \leq \phi(t, x) \leq +\infty \tag{12}$$

Having $f(t, x) > 0$ is a major advantage of the model since in the financial markets forward rates are always positive. In the limit of $\mu \rightarrow 0$ the following HJM-models are covered by eq.(10), and these models have been discussed from an empirical point of view in [13] as follow in table 3.1.

Table.3.1. Previous Formulation on Volatility

| Model | Volatility |
|----------------------------------|--|
| Ho dan Lee (1986) | $\sigma(t, x, f(t, x)) = \sigma_0$ |
| CIR (1985) | $\sigma(t, x, f(t, x)) = \sigma_0 f^{\frac{1}{2}}(t, x)$ |
| Courtadon (1982) | $\sigma(t, x, f(t, x)) = \sigma_0 f(t, x)$ |
| Vasicek (1997) | $\sigma(t, x, f(t, x)) = \sigma_0 \exp(-\lambda(x-t))$ |
| Heath-Jarrow-Morton / HJM (1992) | $\sigma(t, x, f(t, x)) = [\sigma_0 + \sigma_1(x, t), f(t, x)]$ |

How do we generalize the Lagrangian given in eq.(3) to case where the forward rates are always positive? We interpret the Lagrangian given in eq(3) to be an approximate one that valid only if all the forward rates are close to some fixed value f_0 . We then have:

$$\frac{\partial f(t, x)}{\partial t} = f_0 e^{\phi(t,x)} \frac{\partial \phi(t, x)}{\partial t} \tag{13}$$

$$\approx f_0 \frac{\partial \phi(t, x)}{\partial t} + O(\phi^2) \quad (14)$$

Hence we make the following mapping:

$$\frac{\partial f(t, x)}{\partial t} \rightarrow f_0 \frac{\partial \phi(t, x)}{\partial t} \quad (15)$$

Eq(3) then generalizes to:

$$\begin{aligned} \mathcal{L}[\phi] &= \mathcal{L}_{kinetik}[\phi] + \mathcal{L}_{rigiditas}[\phi] \\ &= -\frac{1}{2} \left[\left\{ \frac{f_0 \frac{\partial \phi(t, x)}{\partial t} - \alpha(t, x)}{\sigma_0(t, x) e^{v\phi(t, x)}} \right\}^2 + \frac{1}{\mu^2} \left\{ \frac{\partial}{\partial x} \left(\frac{f_0 \frac{\partial \phi(t, x)}{\partial t} - \alpha(t, x)}{\sigma_0(t, x) e^{v\phi(t, x)}} \right) \right\}^2 \right] \end{aligned} \quad (16)$$

We will show later – in deriving the Hamiltonian – that the system needs a non-trivial integration measure.

We hence define the theory by the Feynman path integral:

$$Z = \int D\phi f^{-v} e^{S[\phi]} \quad (17)$$

$$\int D\phi f^{-v} \equiv \prod_{(t,x) \in P} \int_{-\infty}^{+\infty} d\phi(t, x) f^{-v}(t, x) \quad (18)$$

The boundary conditions given for $f(t, x)$ in eq(5) and (6) continue to hold for stochastic volatility Lagrangian given in eq.(16).

3.2. Volatility as an Independent Quantum Field

We consider the second case where volatility $\sigma(t, x)$ is taken to be an independent (stochastic) quantum field. Since one can only measure the effects of volatility on the forward rates, all the effects of stochastic volatility will be manifested only via the behavior of the forward rates. For simplicity, we consider the forward rate to be a quantum field as given in eq.(4) with:

$$f(t, x): -\infty \leq f(t, x) \leq +\infty \quad (19)$$

Since the volatility function $\sigma(t, x)$ is always positive, that is, $\sigma(t, x) > 0$ we introduce an another quantum field $h(t, x)$ by the following relation (the minus sign is taken for notational convenience):

$$\sigma_0(t, x) = \sigma_0 e^{-h(t, x)}, \quad -\infty \leq h(t, x) \leq +\infty \quad (20)$$

The system now consists of two interacting quantum fields, namely $f(t, x)$ and $h(t, x)$. The interacting system's Lagrangian should have the following features :

- A parameter ξ that quantifies the extent to which the field $h(t, x)$ is non-deterministic. A limit of $\xi \rightarrow 0$ would, in effect, 'freeze' all the fluctuations of the field $h(t, x)$ and reduce it to a deterministic function.
- A parameter κ to control the fluctuations of $h(t, x)$ in the maturity direction similar to the parameter μ that controls the fluctuations of the forward rates $f(t, x)$ in the maturity direction x

- A parameter ρ with: $-1 \leq \rho \leq +1$ that quantifies the correlation of the forward rates' quantum field $f(t, x)$ with the volatility quantum field $h(t, x)$.
- A drift term for volatility, namely $\beta(t, x)$, which is analogous to the drift term $\alpha(t, x)$ for the forward rates.

The Lagrangian for the interacting system is not unique; there is a wide variety of choices that one can make to fulfill all the conditions given above. A possible Lagrangian for the interacting system, written by analogy with the Lagrangian for the case of stochastic volatility for a single security [9], is given by:

$$L = -\frac{1}{2(1-\rho)^2} \left(\frac{\frac{\partial f}{\partial t} - \alpha}{\sigma} - \rho \frac{\frac{\partial h}{\partial t} - \beta}{\xi} \right)^2 - \frac{1}{2} \left(\frac{\frac{\partial h}{\partial t} - \beta}{\xi} \right)^2 - \frac{1}{2\mu^2} \left(\frac{\partial}{\partial x} \left(\frac{\frac{\partial f}{\partial t} - \alpha}{\sigma} \right) \right)^2 - \frac{1}{2\kappa^2} \left(\frac{\partial}{\partial x} \left(\frac{\frac{\partial h}{\partial t} - \beta}{\xi} \right) \right)^2 \tag{21}$$

with action:

$$S[f, h] = \int_{\mathcal{P}} \mathcal{L} \tag{22}$$

We need to specify the boundary conditions for the interacting system. The initial and final conditions for the forward rates $f(t, x)$ given in eq.(5) continue to hold for the interacting case, and are similarly given for the volatility field as the following :

a. Fixed (Dirichlet) Initial and Final Conditions

The initial value is specified from data, that is:

$$T_i(T_f) < x < +T_{FR}, \sigma(T_i, x), \sigma(T_f, x) \tag{23}$$

specified initial and final volatility curves.

The boundary condition in the x direction for the forward rates – as given in eq.(6) – continues to hold – for the interacting case, and for volatility field is similarly given by [10]

b. Free (Neumann) Boundary Conditions

$$T_i < x < T_f ; \frac{\partial}{\partial x} \left(\frac{\partial h(t, x)}{\partial t} - \beta(t, x) \right) = 0 \tag{24}$$

$$; x = t \text{ or } x = t + T_{FR} \tag{25}$$

On quantizing the volatility field $\sigma(t, x)$, the boundary condition for the forward rate $f(t, x)$ given in eq.(6) is rather unusual. On solving the no arbitrage condition, we will find that α is a (quadratic) functional of the volatility field $\sigma(t, x)$; hence the boundary condition eq.(6) is a form of interaction between the $f(t, x)$ and $h(t, x)$ fields.

We need to define the integration measure for the quantum field $h(t, x)$; the derivation of the Hamiltonian for the system dictates the following choice for the measure, namely :

$$\int Df D\sigma^{-1} = \prod_{(t,x) \in \mathcal{P}} \int_{-\infty}^{+\infty} df(t, x) d\sigma^{-1}(t, x) \tag{26}$$

$$\prod_{(t,x) \in \mathcal{P}} \int_{-\infty}^{+\infty} df(t, x) dh(t, x) e^{h(t,x)} \tag{27}$$

The partition function of the quantum field theory for the forward rates with stochastic volatility is defined by Feynman path integral as:

$$Z = \int Df D\sigma^{-1} \tag{28}$$

The (observed) market value of a financial instrument, say $[f, h]$ is expressed as the average value of the instrument – denoted by $\mathcal{O}[f, h]$, – taken over all possible values of the quantum fields $f(t, x)$ and $h(t, x)$ – denoted by $\langle \mathcal{O}[f, h] \rangle$, with the probability density given by the (appropriately normalized) action. In symbols:

$$\langle \mathcal{O}[f, h] \rangle = \frac{1}{Z} \int Df D\sigma^{-1} \mathcal{O}[f, h] e^{S[f, h]} \tag{29}$$

We consider the limit of the volatility being reduced to a deterministic function. For this limit we have ξ, ρ and $\kappa \rightarrow 0$. The kinetic term of the $h(t, x)$ field in the action given in eq.(22) has the limit (up to irrelevant constants)

$$\lim_{\xi \rightarrow 0} \prod_{t,x \in \mathcal{P}} \exp \left\{ -\frac{1}{2} \int_{\mathcal{P}} \left(\frac{\frac{\partial h}{\partial t} - \beta}{\xi} \right)^2 \right\} \rightarrow \prod_{t,x \in \mathcal{P}} \delta \left(\frac{\partial h}{\partial t} - \beta \right) \tag{30}$$

which implies that:

$$\langle \sigma(t, x) \rangle = \sigma_0 \langle e^{-h(t,x)} \rangle \tag{31}$$

$$= \sigma_0 \exp \left\{ - \int_{t_0}^t dt' \beta(t', x) \right\} + \mathcal{O}(\xi, \kappa, \rho) \tag{32}$$

4. State Space Hamiltonian

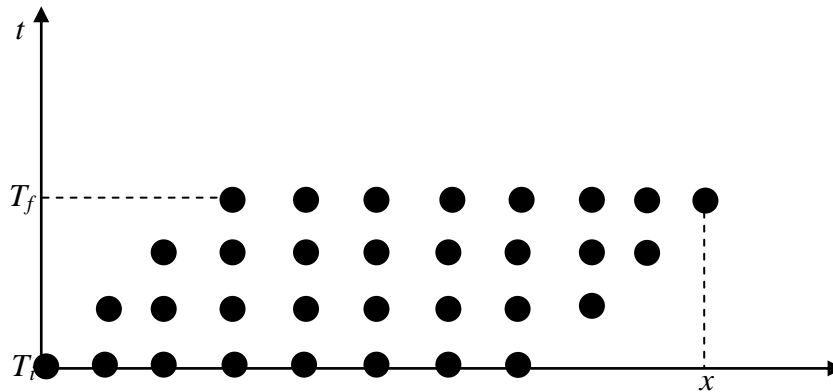
The Feynman path integral formulation given in eq(17) and (28) is useful for calculating the expectation values of quantum fields. To study questions related to the time evolution of quantities of interest, one needs to derive the Hamiltonian for the system from its Lagrangian. Note the route that we are following is opposite to the one taken in [9] where the Lagrangian for a stock price with stochastic volatility was derived starting from its Hamiltonian. The state space of a field theory is a linear vector space – denoted by \mathbf{V} , that consists of functional of the field configurations at some fixed time terdiri atas sejumlah fungsional medan konfigurasi pada waktu terikat t (A brief discussion of the state space is given in [9]). The dual space of \mathbf{V} – denoted by $\mathbf{V}_{\text{rangkap}}$ – consists of all linear mappings from \mathbf{V} to the complex numbers, and is also a linear vector space. Let an element of \mathbf{V} be denoted by $|g\rangle$ and an element of $\mathbf{V}_{\text{rangkap}}$ by $\langle p|$; then $\langle p|g\rangle$ is a complex number. We will refer to both \mathbf{V} and $\mathbf{V}_{\text{rangkap}}$ as the

state space of the system. The Hamiltonian H , is an operator – the quantum analog of energy – that is an element of t tensor product space $\mathcal{V} \otimes \mathcal{V}_{rangkap}$. The matrix elements of are complex numbers, and given by $\langle p|g \rangle$.

In this section, we study the features the state space and Hamiltonian for the forward rates. For notational brevity, we consider the forward rates quantum field $f(t, x)$ to stand for both the quantum fields $f(t, x)$ and $h(t, x)$. Since the Lagrangian for the forward rates given in eq.(21) has only first order derivatives in time, an infinitesimal generator, namely the Hamiltonian H exists for it. Obtaining the Hamiltonian for the forward rates is a complicated exercise due to the non-trivial structure of the underlying domain P . In particular, the forward rates quantum field will be seen to have a distinct state space for every instant t .

For greater clarity, we discrete both time and maturity time into a finite lattice, with lattice spacing in both directions taken to be ϵ (For a string moving with velocity v , the maturity lattice would have spacing of ϵv). On the lattice, the minimum time for futures contract is time ϵ ; for most applications $\epsilon = 1$ day. The points comprising the discrete domain $\tilde{\mathcal{P}}$ are shown in Figure 3.2. :

Gambar 3.2. Lattice in Time and Maturity Directions in domain $\tilde{\mathcal{P}}$



The discrete domain $\tilde{\mathcal{P}}$ is given by:

$$(t, x) \rightarrow \epsilon(n, l) \text{ with } n, l \text{ integers} \tag{33}$$

$$(T_i, T_f, T_{FR}) \rightarrow \epsilon(N_i, N_f, N_{FR}) \tag{34}$$

$$\text{lattice } \tilde{\mathcal{P}} = \{(n, l) | N_i \leq n \leq N_f : n \leq l \leq (n + N_{FR})\} \tag{35}$$

$$f(t, x) \rightarrow f_{n,l} \tag{36}$$

$$\frac{\partial f(t, x)}{\partial t} \approx \frac{f_{n+1,l} - f_{n,l}}{\epsilon} ; \frac{\partial f(t, x)}{\partial x} \approx \frac{f_{n,l+1} - f_{n,l}}{\epsilon} \tag{37}$$

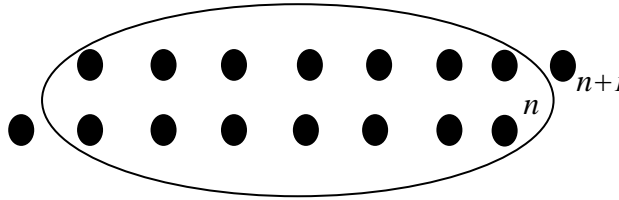
The partition function is now given by a finite multiple integral, namely:

$$Z = \prod_{(n,l) \in \tilde{\mathcal{P}}} \int df_{n,l} e^{S[f]} \quad (38)$$

$$S = \sum_n S(n) \quad (39)$$

Consider two adjacent time slices labeled by n and $n+1$, as shown in Figure 3.3 dibawah ini. $S(n)$ is the action connecting the forward rates of these two time slices.

Figure 3.3. Two Consecutive Time Slices for $t = n\varepsilon$ and $t = (n+1)\varepsilon$



As can be seen from Figure 4, for the two time slices there is a mismatch of the 2-lattice sites on the edges, namely, lattice sites (n, n) at time n and $(n+1, n+1+N_{FR})$ at time $n+1$ are not in common. We isolate the unmatched variables and have the following :

Variables at time n :

$$\{f_{n,n}, \tilde{F}_n\}; \quad \tilde{F}_n \equiv \{f_{n+1,l} | n \leq l \leq n + N_{FR}\} \quad (40)$$

Variables at time $(n+1)$:

$$\{F_n, f_{n+1, n+1+N_{FR}}\}; \quad F_n \equiv \{f_{n+1,l} | n+1 \leq l \leq n+1 + N_{FR}\} \quad (41)$$

Note that although the variables F_n refer to time $(n+1)$, we label it with earlier time n for later convenience. From Figure 3.3 we see that both sets of variables F_n and \tilde{F}_n cover the same lattice sites in the maturity direction x namely $n+1 \leq l \leq n + N_{FR}$, and hence have the same number of forward rates, namely $N_{FR} - 1$. The Hamiltonian will be expressible solely in terms of these variables.

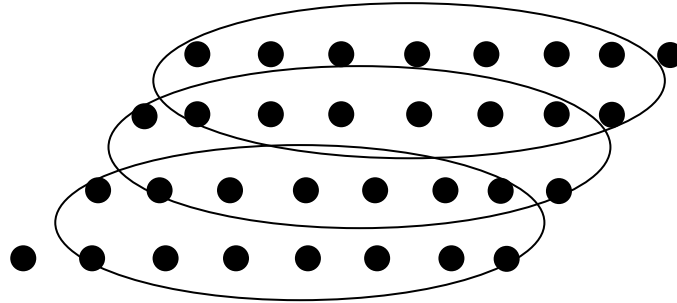
From the discretized time derivatives defined in eq.(37) the discretized action $S(n)$ contains terms that couple only the common points in the lattice for the two time slices, namely the variables belonging to the sets $\tilde{F}_n; F_n$. We hence have for the action:

$$S(n) = \in \sum_{\{l\}} \mathcal{L}_n [f_{n,l}, f_{n+1,l}] \quad (42)$$

$$S(n) = \in \sum_{\{l\}} \mathcal{L}_n [\tilde{F}_n; F_n] \quad (43)$$

As shown is in Figure 3.4, the action for the entire domain $\tilde{\mathcal{P}}$ shown in Figure 3.2 can be constructed by repeating the construction given in Figure 3.3 and summing over the action $S(n)$ over all time $N_i \leq n \leq N_f$

Picture of 3.4. Reconstructing the Lattice from the Two Time Slices



The Hamiltonian of the forward rates is an operator that acts on the state space of states of the forward rates; we hence need to determine the co-ordinates of its state space.

Consider again the two consecutive time slices n and $n+1$ given in Figure 3.4. We interpret the forward rates for two adjacent instants, namely, $\{f_{n,n}, \tilde{F}_n\}$ dan $\{F_n, f_{n+1, n+1+N_{FR}}\}$ given in eq.(40) – and which appear in the action eq.(42) – as the co-ordinates of the state spaces \mathbf{V} and \mathbf{V}_{dual} respectively.

For every instant of time n there is a distinct state space \mathcal{V}_n dan $\mathcal{V}_{\text{rangkap}, n}$. The co-ordinates of the state spaces \mathcal{V}_n dan \mathcal{V}_{n+1} are given by the tensor product of the space of state for every maturity point l namely :

$$\langle \tilde{f}_n | = \bigotimes_{n \leq l \leq n+N_{FR}} \langle f_{n,l} | \equiv \langle f_{n,n} | \langle \tilde{F}_n | \quad (44)$$

coordinate state for $\mathcal{V}_{\text{rangkap}, n}$

$$| f_{n+1} \rangle = \bigotimes_{n+1 \leq l \leq n+1+N_{FR}} | f_{n+1,l} \rangle \equiv | F_n \rangle | f_{n, n+1+N_{FR}} \rangle \quad (45)$$

coordinate state for \mathcal{V}_{n+1}

The state vector $| F_n \rangle$ belongs to the space $| F_n \rangle$ but we reinterpret $| F_n \rangle$ as corresponding to the state space \mathcal{F}_n at earlier time n . This interpretation allows us to study the system instantaneously using the Hamiltonian formalism consists of all possible functions of N_R forward rates $\{f_{n,n}, \tilde{F}_n\}$. The state spaces differ for greater clarity, we discrete both time and maturity time into a finite lattice, with lattice spacing in both directions taken to be ϵ (For a string moving with velocity v , the maturity lattice would have spacing of ϵv). On the lattice, the minimum time for futures contract is time ϵ ; for most applications $\epsilon = 1$ day. The points comprising the discrete domain $\tilde{\mathcal{P}}$ are shown in Figure 3.2. :

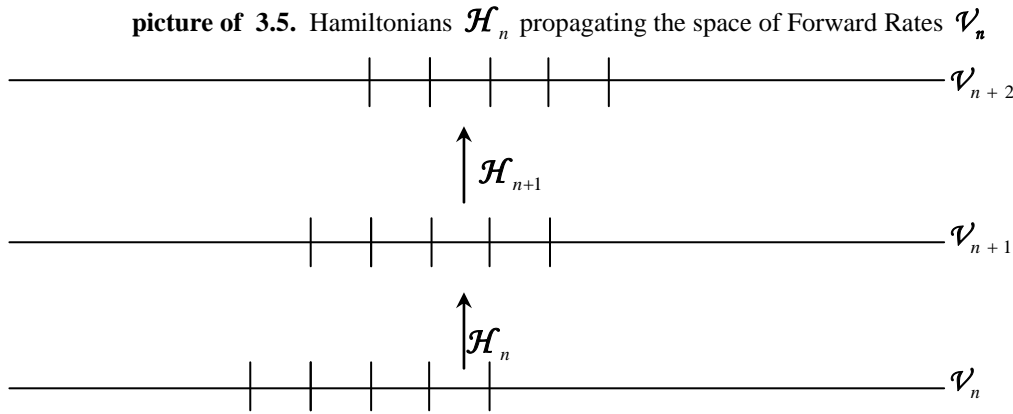
r for different n by the fact that a different set of forward rates comprise its set of independent variables. Although the state spaces \mathcal{V}_n and \mathcal{V}_{n+1} are not the identical, there is an intersection of these two spaces, namely $\mathcal{V}_n \cap \mathcal{V}_{n+1}$ that covers the same interval in the maturity direction, and is coupled by the action $S(n)$. The intersection yields a state space, namely \mathcal{F}_n on which the Hamiltonian evolution of the forward rates takes place. In symbols, we have :

$$\mathcal{V}_{n+1} = \mathcal{F}_n \otimes | f_{n+1, n+1+N_{FR}} \rangle \quad (46)$$

$$\mathcal{V}_{rangkap,n} = \langle f_{n,n} | \otimes \mathbf{F}_{rangkap,n} \tag{47}$$

$$\mathcal{H}_n : \mathbf{F}_n \rightarrow \mathbf{F}_n \Rightarrow H_n \in \mathcal{V}_{rangkap,n} \otimes \mathcal{V}_{n+1} \tag{48}$$

The Hamiltonian \mathcal{H}_n is an element on the tensor product space spanned by the operators $|F_n\rangle\langle\tilde{F}_n|$, namely the space of operators given by $\mathbf{F}_n \otimes \mathbf{F}_{dual,n}$. The vector spaces \mathcal{V}_n and the Hamiltonian \mathcal{H}_n acting on these spaces is shown in Figure 3.5



Note that both the states $||F_n\rangle\rangle$ dan $|\tilde{F}_n\rangle$ belong to the same state space \mathbf{F}_n , and we use twiddle to indicate that the two states are different; in contrast, for example, the two states $|f\rangle$ and $\langle f|$ indicate that one state is the dual of the other.

As one scans through all possible values for the forward rates and \hat{f} , one obtains a complete basis for the state space \mathcal{V}_n . In particular, the resolution of the identity operator for \mathcal{V}_n , denoted by I_n is a reflection that the basis states are complete, and is given by [9] :

$$I_n = \prod_{n \leq l \leq n+N_{FR}} \int df_{n,l} |f_n\rangle\langle f_n| \tag{49}$$

$$\equiv \int df_{n,n} \cdot d\tilde{F}_n |f_{n,n}; \tilde{F}_n\rangle\langle f_{n,n}; \tilde{F}_n| \tag{50}$$

The Hamiltonian of the system \mathcal{H} is defined by the Feynman formula (up to a normalization), from eq.(42), has:

$$\rho_n e^{\in \sum_{\{l\}} \mathcal{L}_n[f_{n,l}, f_{n+1+l}]} = \langle f_{n,n}, \tilde{F}_n | e^{-\in \mathcal{H}_n} | F_n, f_{n+1, n+1+N_{FR}} \rangle \tag{51}$$

where in general ρ_n is a field-dependent measure term. Using the property of the discrete action given in eq.(43), we have :

$$\rho_n e^{\sum_{\{t\}} \mathcal{L}_n[F_n, \tilde{F}_n]} = \langle f_{n,n}, \tilde{F}_n | e^{-\epsilon \mathcal{H}_n} | F_n, f_{n+1, n+1+N_{FR}} \rangle \quad (52)$$

$$= \langle \tilde{F}_n | e^{-\epsilon \mathcal{H}_n} | F_n \rangle \quad (53)$$

Equation (53) is the main result of this Section.

In going from eq.(52) to eq.(53) we have used the fact that the action connecting time slices n and $n+1$ does not contain the variables $f_{n,n}$ and $f_{n+1+N_{FR}}$ respectively. This leads to the result that the Hamiltonian consequently does not depend on these variables. The interpretation of eq.(53) is that the Hamiltonian propagates the initial state $\langle \tilde{F}_n |$ in time ϵ to the final state $| F_n \rangle$. Note the relation:

$$\langle f_{n,n}, \tilde{F}_n | e^{-\epsilon \mathcal{H}_n} | F_n, f_{n+1, n+1+N_{FR}} \rangle = \langle \tilde{F}_n | e^{-\epsilon \mathcal{H}_n} | F_n \rangle \quad (54)$$

shows that there is an asymmetry in the time direction, with the Hamiltonian being independent of the earliest forward rate f_n , n of the initial state and of the latest forward rate $f_{n,n}$, $f_{n+1+N_{FR}}$ of the final state. It is this asymmetry in the propagation of the forward rates which yields the parallelogram domain $\tilde{\mathcal{P}}$ given in Figure 3.2, and reflects the asymmetry that the forward rates $f(t, x)$ exist only for $x > t$.

For notational simplicity, we henceforth use continuum notation; in particular, the state space is labelled by \mathcal{V}_t , and state vector by $| f_t \rangle$. The elements of the state space of the forward rates \mathcal{V}_t includes all the financial instruments that are traded in the market at time t . In continuum notation from eq.(45), we have that:

$$| f_t \rangle = \bigotimes_{t \leq x \leq t+T_{FR}} | f(t, x) \rangle \quad (55)$$

$$| F_t \rangle = \bigotimes_{t \leq x \leq t+T_{FR}} | f(t, x) \rangle \quad (56)$$

In continuum notation, the only difference between state vectors $| f_t \rangle$ and $| F_t \rangle$ is that, in eq.(56), the point $x = t$ is excluded in the continuous tensor product.

The partition function Z given in eq.(38) can be reconstructed from the Hamiltonian by recursively applying the procedure discussed for the two time slices. We then have, in continuum notation, that :

$$Z = \int D f e^{S[f]} \quad (57)$$

$$= \langle f_{initial} | \mathcal{T} \left\{ \exp \left(- \int_{T_i}^{T_f} \mathcal{H}_t dt \right) \right\} | f_{final} \rangle \quad (58)$$

where the symbol \mathcal{T} in the equation above stands for time ordering the (non- commuting) operators in the argument, with the earliest time being placed to the left.

5. Conclusion

We made a generalization of the field theory model for the forward rates to account for stochastic volatility by treating volatility either as a function of the forward rates or as an independent quantum field. In both cases, the Feynman path integral could be naturally extended to account for stochastic volatility. For the case of deterministic volatility, it was found in [10] that in effect the two dimensional quantum field theory reduced to a one-dimensional problem due to the specific nature of the Lagrangian. However, on treating volatility as a quantum field, the theory is now irreducibly two-dimensional, and displays all the features of a quantum field theory.

To obtain the Hamiltonian of the forward rates, we were in turn led to an analysis of the underlying state space of the system, which turned out to be non-trivial due to the parallelogram domain on which the forward rates are defined. The Hamiltonian for the forward rates is an independent formulation of the theory of the forward rates, and can lead to new insights on the behaviour of the forward rates. The model for the forward rates with stochastic volatility has a number of **free parameters** that can only be determined by studying the market. Hence one needs to be numerically analyze the model so as to calibrate it, and to test its ability to explain the market's behaviour. The first step in this direction has been taken in [11] and these calculations are now being extended to the case of stochastic volatility.

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