FUNGSI KHUSUS DALAM BENTUK INTEGRAL

FUNGSI FAKTORIAL

Definisi

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

Buktikan bahwa : $0! = 1$

$$0! = \int_0^\infty x^0 e^{-x}\,dx = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = -(0-1) = 1$$

Terbukti

FUNGSI Gamma

Definisi

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx \,; \qquad p > 0$$

Hubungan fungsi Gamma dengan fungsi Faktorial

$$\Gamma(p+1) = \int_0^\infty x^{(p-1)+1} e^{-x}\, dx = \int_0^\infty x^{p} e^{-x}\, dx = p!$$

$$\Gamma(p+1) = p!$$

$$\Gamma(1) = 0! = 1, \qquad \Gamma(2) = 1! = 1, \qquad \Gamma(3) = 2! = 2 \qquad dst$$

Nilai Fungsi Gamma ditabulasi untuk Gamma 1 sampai dengan Gamma 2

Hubungan Rekursif Fungsi Gamma

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x}\, dx$$

Lakukan integrasi parsial, seperti berikut :

$$u = x^p \qquad du = px^{p-1}\, dp$$

$$dv = e^{-x}\, dx \qquad v = -e^{-x}$$

$$\Gamma(p+1) = -x^p e^{-x}\Big|_0^\infty - \int_0^\infty \left(-e^{-x}\right) px^{p-1}\, dx$$

$$\Gamma(p+1) = p\int_0^\infty x^{p-1} e^{-x}\, dx = p\Gamma(p)$$

$$\boxed{\Gamma(p+1) = p\Gamma(p)}$$

Hubungan Rekursif Fungsi Gamma

$$\boxed{\Gamma(p+1) = p\Gamma(p)}$$

$$\Gamma(3)=\Gamma(2+1)= 2\Gamma(2)= 2(1) = 2$$

$$\Gamma(4)= \Gamma(3+1)=3\Gamma(3)=3\Gamma(2+1)=3.2\Gamma(2)=6(1)=6$$

dst

Hubungan Rekursif Fungsi Gamma

$$\Gamma(p) = \frac{1}{p}\Gamma(p+1)$$

$$\Gamma(0{,}6) = \frac{1}{0{,}6}\Gamma(0{,}6+1) = \frac{1}{0{,}6}\Gamma(1{,}6) \longrightarrow$$ **Tabel F. Gamma**

$$\Gamma(-1{,}5) = \frac{1}{-1{,}5}\Gamma(-1{,}5+1) = \frac{1}{-1{,}5}\Gamma(-0{,}5)$$

$$= \frac{1}{-1{,}5}\frac{1}{-0{,}5}(-0{,}5+1)$$

$$= \frac{1}{-1{,}5}\frac{1}{-0{,}5}\Gamma(0{,}5)$$

$$= \frac{1}{-1{,}5}\frac{1}{-0{,}5}\frac{1}{0{,}5}\Gamma(1{,}5) \longrightarrow$$ **Tabel F. Gamma**

Nilai Γ(0,5)

$$\Gamma(0{,}5) = \int_0^\infty t^{0{,}5-1} e^{-t}\, dt = \int_0^\infty \frac{1}{\sqrt{t}} e^{-t}\, dt$$

Misalkan t = y², maka dt = 2y dy

$$\Gamma(0{,}5) = 2\int_0^\infty \frac{1}{y} e^{-y^2}\, y dy$$

$$\Gamma(0{,}5) = 2\int_0^\infty \frac{1}{x} e^{-x^2}\, x dx$$

$$[\Gamma(0{,}5)]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$$

Nilai $\Gamma(0{,}5)$

$$\left[\Gamma(0{,}5)\right]^2 = 4\int_{\theta=0}^{\pi/2}\int_{0}^{\infty} e^{-r^2}\, r dr\, d\theta = 4\,\frac{\pi}{2}\,\frac{e^{-r^2}}{-2}\Bigg|_{0}^{\infty} = \pi$$

$$\left[\Gamma(0{,}5)\right]^2 = \pi$$

$$\Gamma(0{,}5) = \sqrt{\pi}$$

Contoh soal

Hitunglah integral berikut dengan Fungsi Gamma :

$$\int_0^\infty x^2 e^{-x^2}\, dx$$

Jawab

$$\int_0^\infty x^2 e^{-x^2}\, dx$$

Misal $u = x^2$, maka $du = 2x\, dx$

Untuk $x = 0$ maka $u = 0$

Untuk $x = \infty$ maka $u = \infty$

$$\int_0^\infty u e^{-u} \frac{du}{2\sqrt{u}} = \frac{1}{2}\int_0^\infty u^{1/2} e^{-u}\, du = \frac{1}{2}\Gamma(1,5) = \frac{1}{2}\frac{1}{2}\Gamma(0,5) = \frac{\sqrt{\pi}}{4}$$

$$\int_0^\infty x^2 e^{-x^2}\, dx = \frac{\sqrt{\pi}}{4}$$

Soal Latihan

Hitunglah integral-integral berikut dengan Fungsi Gamma :

1. $\displaystyle\int_0^{\infty} \sqrt{t}\,e^{-3t}\,dt$

2. $\displaystyle\int_0^{\infty} \left(2y^2 + y\right)e^{-y^2}\,dy$

3. $\displaystyle\int_0^{1} \left(\ln x\right)^{1/3}\,dx$

Formula penting terkait Fungsi Gamma

$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin \pi p}$$

Untuk p = 0,5

$$\Gamma(0,5)\,\Gamma(1-0,5) = \frac{\pi}{\sin 0,5\pi}$$

$$\Gamma(0,5)\,\Gamma(0,5) = \frac{\pi}{1}$$

$$[\Gamma(0,5)]^2 = \pi$$

$$\Gamma(0,5) = \sqrt{\pi}$$

Contoh soal

Buktikan bahwa :

$$z!(-z)! = \frac{\pi z}{\sin \pi z}$$

Bukti

$$z!(-z)! = \Gamma(z+1)\,\Gamma(1-z)$$

$$z!(-z)! = z\Gamma(z)\,\Gamma(1-z)$$

$$z!(-z)! = z[\Gamma(z)\Gamma(1-z)]$$

$$z!(-z)! = z\left[\frac{\pi}{\sin \pi z}\right]$$

$$z!(-z)! = \frac{\pi z}{\sin \pi z}$$

Ingat :

$$\Gamma(n+1) = n!$$

dan

$$\Gamma(n+1) = n\Gamma(n)$$

dan

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin \pi n}$$

Fungsi Beta

Definisi 1 :

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx\,; \qquad p>0;\ q>0.$$

$$B(p,q)=B(q,p)$$

Definisi 2 :

$$B(p,q)=\frac{1}{a^{p+q-1}}\int_0^a y^{p-1}(a-y)^{q-1}\,dy$$

Fungsi Beta

Definisi 3 :

$$B(p,q) = 2\int_{0}^{\pi/2} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1}\, d\theta$$

Definisi 4 :

$$B(p,q) = \int_{0}^{\infty} \frac{y^{p-1}\, dy}{(1+y)^{p+q}}$$

Hubungan Fungsi Beta dan Fungsi Gamma

Fungsi Gamma

$$\Gamma(p) = \int_0^{\infty} t^{p-1} e^{-t}\, dt$$

Misal t = y², maka

$$\Gamma(p) = 2\int_0^{\infty} y^{2p-1} e^{-y^2}\, dy \quad \textbf{atau} \quad \Gamma(q) = 2\int_0^{\infty} x^{2q-1} e^{-x^2}\, dx$$

Jika Γ(p) dikalikan dengan Γ(q) dikalikan maka :

$$\Gamma(p)\Gamma(q) = 4\int_0^{\infty}\int_0^{\infty} x^{2q-1} y^{2p-1} e^{-\left(x^2+y^2\right)}\, dx\, dy$$

$$\Gamma(p)\Gamma(q) = 4\int_0^{\infty}\int_0^{\pi/2} (r\cos\theta)^{2q-1} (r\sin\theta)^{2p-1} e^{-r^2}\, r\,dr\,d\theta$$

Hubungan Fungsi Beta dan Fungsi Gamma

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty r^{2p+2q-1}e^{-r^2}\,dr\int_0^{\pi/2}(\cos\theta)^{2q-1}(\sin\theta)^{2p-1}\,d\theta$$

$$\int_0^\infty r^{2p+2q-1}e^{-r^2}\,dr \;\rightarrow\; \frac{1}{2}\Gamma(p+q)$$

$$\int_0^{\pi/2}(\cos\theta)^{2q-1}(\sin\theta)^{2p-1}\,d\theta \;\rightarrow\; \frac{1}{2}B(p,q)$$

$$\Gamma(p)\Gamma(q) = 4\frac{1}{2}\Gamma(p+q).\frac{1}{2}B(p,q)$$

$$\boxed{B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}}$$

Hubungan Fungsi Beta dan Fungsi Gamma

contoh

$$B\left(1, \tfrac{1}{2}\right) = \frac{\Gamma(1)\Gamma\left(\tfrac{1}{2}\right)}{\Gamma\left(\tfrac{3}{2}\right)} = \frac{(1)\left(\sqrt{\pi}\right)}{\tfrac{1}{2}\sqrt{\pi}} = 2$$

Contoh soal 1

Selesaikan integral berikut dengan fungsi Beta

$$\int_0^{\pi/2} \sqrt{\sin^3 x \cos x}\, dx$$

Solusi dengan def. Fungsi Beta 3

$$B(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1}\, d\theta$$

$$\int_0^{\pi/2} \left(\sin^3 x \cos x\right)^{1/2} dx = \int_0^{\pi/2} (\sin x)^{3/2} (\cos x)^{1/2}\, dx$$

$$2p - 1 = 3/2 \qquad p = 5/4 \qquad\qquad 2q - 1 = 1/2 \qquad q = 3/4$$

$$\int_0^{\pi/2} \sqrt{\sin^3 x \cos x}\, dx = \frac{1}{2} B\left(5/4, 3/4\right) = \frac{1}{2} \frac{\Gamma\left(5/4\right)\Gamma\left(3/4\right)}{\Gamma(2)} = \ldots\ldots$$

Contoh soal 2

Selesaikan integral berikut dengan fungsi Beta

$$\int_0^\infty \frac{y^2\,dy}{(1+y)^6}$$

Solusi dengan def. Fungsi Beta 4

$$B(p,q) = \int_0^\infty \frac{y^{p-1}\,dy}{(1+y)^{p+q}}$$

$$p-1=2 \qquad p=3 \qquad\qquad (p+q)=6 \qquad q=3$$

Jadi

$$\int_0^\infty \frac{y^2\,dy}{(1+y)^6} = B(3,3) = \frac{\Gamma(3)\Gamma(3)}{\Gamma(6)} = \frac{4}{120}$$

Soal Latihan

Selesaikan integral berikut dengan fungsi Beta yang sesuai

1. $$\int_0^\infty \frac{y\,dy}{\left(1+y^3\right)^2}$$

2. $$\int_0^2 \frac{x^2 dx}{\sqrt{2-x}}$$

3. $$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}$$

Aplikasi dalam persoalan Fisika

Sket grafik $x^2 + y^2 = 9$

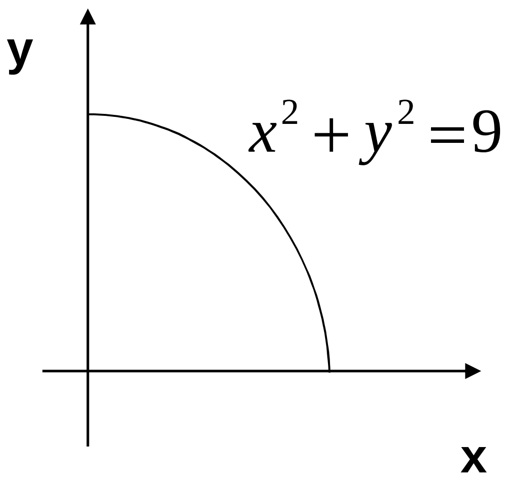
Gunakan fungsi Beta untuk menghitung :

a. Luas daerah pada kuadran pertama

b. Titik pusat massa dari daerah ini (anggap rapat massanya seragam)

Jawab

a



$$A = \int dA = \iint dx\,dy = \int_0^3 \int_0^{\sqrt{9-y^2}} dx\,dy$$

$$A = \int_0^3 \sqrt{9 - y^2}\,dy$$

Aplikasi dalam persoalan Fisika

$$A=\int_0^3 \sqrt{9-y^2}\,dy$$

Misalkan : $y^2 = x \longrightarrow 2y\,dy = dx$

Batas : $y=0 \rightarrow x=0$

$y=3 \rightarrow x=9$

$$A=\int_0^9 \sqrt{9-x}\,\frac{dx}{2\sqrt{x}} = \frac{1}{2}\int_0^9 x^{-1/2}(9-x)^{1/2}\,dx$$

Gunakan Fungsi Beta kedua

$$B(p,q) = \frac{1}{a^{p+q-1}}\int_0^a y^{p-1}(a-y)^{q-1}\,dy$$

$$\int_0^a y^{p-1}(a-y)^{q-1}\,dy = a^{p+q-1}B(p,q)$$

$$A = \frac{1}{2}\,(9)^{1/2+3/2-1}\;B(1/2, 3/2)$$

$$A = \frac{1}{2}\,(9)\;\frac{\Gamma(1/2)\Gamma(3/2)}{\Gamma(2)}$$

$$A = \frac{1}{2}\,(9)\;\frac{\sqrt{\pi}\;\;1/2\sqrt{\pi}}{1} = \frac{9\pi}{4}$$

Dengan rumus luas lingkaran :

$$A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(3)^2 = \frac{9\pi}{4}$$

Sama

B. Titik pusat massa luasan tersebut dihitung dengan rumus :

$$x_{pm} = \frac{\int x \, dM}{\int dM} = \frac{\int \rho \, x dA}{\int \rho dA}$$

$$y_{pm} = \frac{\int y \, dM}{\int dM} = \frac{\int \rho \, y dA}{\int \rho dA}$$

Dst ....

Fungsi Error dan Pelengkapnya

Definisi Fungsi Error

$$Erf\left(x\right) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2}\, dt$$

Definisi Pelengkap Fungsi Error

$$Erfc\left(x\right) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2}\, dt$$

Fungsi Error

$$Erf(\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt$$

Selesaikan dengan Fungsi Gamma

$$Erf(\infty) = \frac{2}{\sqrt{\pi}} \frac{1}{2} \Gamma(1/2) = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

Fungsi Error dan Pelengkapnya

$$Erf(x) + Erfc(x) = \frac{2}{\sqrt{\pi}}\left(\int_0^x e^{-t^2}\,dt + \int_x^\infty e^{-t^2}\,dt\right)$$

$$Erf(x) + Erfc(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt$$

$$Erf(x) + Erfc(x) = Erf(\infty) = 1$$

$$\boxed{Erf(x) = 1 - Erfc(x)} \qquad \boxed{Erfc(x) = 1 - Erf(x)}$$

Fungsi Error

Dari deret pangkat tak hingga :

$$e^{t} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots \quad \Rightarrow \quad e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} + \ldots$$

$$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \left(1 - t^2 + \frac{t^4}{2!} - \ldots\ldots \right) dt$$

$$Erf(x) = \frac{2}{\sqrt{\pi}} \left(t - \frac{t^3}{3} + \frac{t^5}{5.2!} - \ldots\ldots \right)_0^x$$

$$\boxed{Erf(x) = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \ldots\ldots \right)} \qquad |x| << 1$$

Pelengkap Fungsi Error

Definisi pelengkap fungsi error

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt.$$

Kita tuliskan :

$$e^{-t^2} = \frac{1}{t}\, te^{-t^2} = \frac{1}{t}\, \frac{d}{dt}\left(-\tfrac{1}{2}e^{-t^2}\right)$$

dan lakukan integrasi by part sbb :

Pelengkap Fungsi Error

$$\int_x^\infty e^{-t^2}\,dt = \int_x^\infty \frac{1}{t}\frac{d}{dt}\left(-\tfrac{1}{2}e^{-t^2}\right)dt$$

$$= \frac{1}{t}\left(-\tfrac{1}{2}e^{-t^2}\right)\Big|_x^\infty - \int_x^\infty \left(-\tfrac{1}{2}e^{-t^2}\right)\left(-\frac{1}{t^2}\right)dt$$

$$= \frac{1}{2x}e^{-x^2} - \frac{1}{2}\int_x^\infty \frac{1}{t^2}e^{-t^2}\,dt.$$

Sekarang tuliskan lagi :

$$\frac{1}{t^2}e^{-t^2} = \left(\frac{1}{t^3}\right)\left(\frac{d}{dt}\right)\left(-\frac{1}{2}e^{-t^2}\right)$$

kemudian lakukan integral by part lagi sbb :

Pelengkap Fungsi Error

$$\int_x^\infty \frac{1}{t^2} e^{-t^2}\, dt = \int_x^\infty \frac{1}{t^3} \frac{d}{dt}\left(-\tfrac{1}{2}e^{-t^2}\right) dt$$

$$= \frac{1}{t^3}\left(-\tfrac{1}{2}e^{-t^2}\right)\Big|_x^\infty - \int_x^\infty \left(-\tfrac{1}{2}e^{-t^2}\right)\left(-\frac{3}{t^4}\right) dt$$

$$= \frac{1}{2x^3} e^{-x^2} - \frac{3}{2}\int_x^\infty \frac{1}{t^4} e^{-t^2}\, dt.$$

Jika proses ini terus dilanjutkan, maka akan didapat ungkapan deret untuk pelengkap fungsi error, sbb :

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}}\left(1 - \frac{1}{2x^2} + \frac{1\cdot 3}{(2x^2)^2} - \frac{1\cdot 3\cdot 5}{(2x^2)^3} + \cdots\right).$$

$$|x| >> 1$$

Contoh soal

1. $$\int_0^2 e^{-x^2}\,dx$$

2. $$\frac{2}{\sqrt{\pi}}\int_5^{10} e^{-u^2}\,du$$

3. $$\sqrt{\frac{2}{\pi}}\int_1^{\infty} e^{-t^2/2}\,dt$$

Formula Stirling

Formula Stirling adalah formula pendekatan untuk fungsi Faktorial dan Fungsi Gamma, sbb :

$$n! \sim n^n e^{-n}\sqrt{2\pi n} \qquad \text{or} \qquad \Gamma(p+1) \sim p^p e^{-p}\sqrt{2\pi p}.$$

Bukti

$$\Gamma(p+1) = p! = \int_0^\infty x^p e^{-x}\,dx = \int_0^\infty e^{p\ln x - x}\,dx.$$

Substitusi variabel baru y sehingga $x = p + y\sqrt{p}.$

$$dx = \sqrt{p}\,dy,$$

Formula Stirling

persamaan di atas menjadi :

$$p! = \int_{-\sqrt{p}}^{\infty} e^{p \ln(p + y\sqrt{p}) - p - y\sqrt{p}} \sqrt{p}\, dy.$$

untuk p besar, logaritma dapat diekspansi dalam deret pangkat berikut :

$$\ln(p + y\sqrt{p}) = \ln p + \ln\left(1 + \frac{y}{\sqrt{p}}\right) = \ln p + \frac{y}{\sqrt{p}} - \frac{y^2}{2p} + \cdots.$$

sehingga

Formula Stirling

$$p! \sim \int_{-\sqrt{p}}^{\infty} e^{p \ln p + y\sqrt{p} - (y^2/2) - p - y\sqrt{p}} \sqrt{p}\, dy$$

$$= e^{p \ln p - p} \sqrt{p} \int_{-\sqrt{p}}^{\infty} e^{-y^2/2}\, dy$$

$$= p^p e^{-p} \sqrt{p} \left[\int_{-\infty}^{\infty} e^{-y^2/2}\, dy - \int_{-\infty}^{-\sqrt{p}} e^{-y^2/2}\, dy \right].$$

$$\sqrt{2\pi} \qquad\qquad 0$$

Untuk p besar

$$p! \sim p^p e^{-p} \sqrt{2\pi p}$$

Bandingkan nilai eksak n! dengan formula pendekatan Stirling

n	n! eksak	n! Formula Stirling	Persen selisih
5	120		
20	$2,43 \times 10^{18}$		
50	$3,04 \times 10^{64}$		

Soal

1. Dalam mekanika statistik sering digunakan persamaan :

$$\ln N! = N \ln N - N$$

disini N berorde bilangan Avogadro, $N = 10^{26}$

Buktikan dari Formula Stirling !

2. Hitunglah :

$$\lim_{n\to\infty} \frac{(2n)!\sqrt{n}}{2^{2n}(n!)^2}$$

3. Hitunglah : $\Gamma(55,5)$

Integral Eliptik

Bentuk Legendre

disebut juga integral eliptik tak lengkap jenis ke satu dan ke dua

$$F(k, \phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \qquad \left\{ \begin{array}{l} 0 \le k \le 1, \\ \qquad \text{or} \\ k = \sin \theta, \quad 0 \le \theta \le \dfrac{\pi}{2}. \end{array} \right.$$

$$E(k, \phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \phi}\, d\phi,$$

k disebut modulus dan ϕ disebut amplitudo integral eliptik. Integral ini ditabulasi untuk nilai θ = arc sin k dan ϕ antara 0 dan $\pi/2$.

k^2 dapat dilihat dari bentuk integral, dengan mengetahui k maka θ dapat ditentukan, sedangkan ϕ dapat dilihat pada batas integral, dengan mengetahui θ dan ϕ, maka nilai integral eliptik dapat dilihat pada tabel integral eliptik F(k,ϕ) dan E(k,ϕ).

Integral Eliptik

Integral eliptik lengkap

Integral eliptik lengkap jenis pertama dan kedua adalah nilai-nilai K dan E (sebagai fungsi k) untuk $\phi = \pi/2$

$$K \quad \text{or} \quad K(k) = F\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

$$E \quad \text{or} \quad E(k) = E\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \phi}\, d\phi.$$

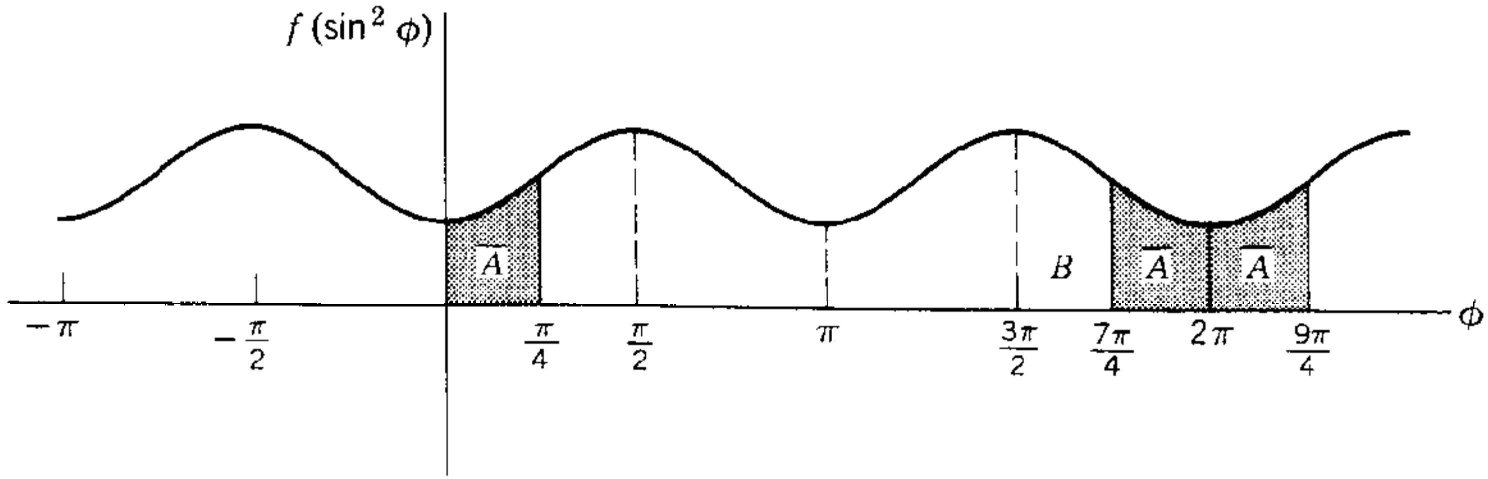
Sama seperti sebelumnya, k^2 dapat dilihat dari bentuk integral, dengan mengetahui k maka θ dapat ditentukan, dengan mengetahui θ, maka nilai integral eliptik lengkap dapat dilihat pada tabel integral eliptik lengkap K dan E

Contoh soal

1. $\displaystyle\int_0^{\pi/4} \frac{d\phi}{\sqrt{1-0,25\sin^2\phi}}$

Bagaimana menghitung integral eliptik untuk $\phi > \pi/2$???

Tinjau fungsi $\sin^2 x$ [$f(\sin^2 x)$] yang merupakan integran dari integral eliptik



$$\int_0^{9\pi/4} \ldots\ldots = \int_0^{2\pi} \ldots\ldots + luas\ A = \int_0^{2\pi} \ldots + \int_0^{\pi/4} \ldots. = 4\int_0^{\pi/2} \ldots. + \int_0^{\pi/4} \ldots$$

$$\int_0^{7\pi/4} \ldots\ldots = \int_0^{2\pi} \ldots\ldots - luas\ A = \int_0^{2\pi} \ldots - \int_0^{\pi/4} \ldots. = 4\int_0^{\pi/2} \ldots. - \int_0^{\pi/4} \ldots$$

catat

$$\int_0^{7\pi/4} \ldots\ldots \neq \int_0^{3\pi/2} \ldots\ldots + luas\ A = \int_0^{3\pi/2} \ldots + \int_0^{\pi/4} \ldots. = 3\int_0^{\pi/2} \ldots. + \int_0^{\pi/4} \ldots$$

Contoh soal

$$\int_0^{5\pi/4} \sqrt{1-0{,}037\sin^2\phi}\; d\phi$$

Jika batas bawah integral tidak nol, maka :

$$\int_{\phi_1}^{\phi_2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \int_0^{\phi_2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} - \int_0^{\phi_1} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

$$= F(k, \phi_2) - F(k, \phi_1),$$

dan jika salah satu batas integral adalah negatif, maka :

$$F(k, -\phi) = \int_0^{-\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = -\int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = -F(k, \phi)$$

Karena $F(k,\phi)$ dan $E(k,\phi)$ merupakan fungsi ganjil

Contoh soal

$$\int_{-7\pi/8}^{11\pi/4} \sqrt{1-0{,}64\sin^2\phi}\; d\phi$$

Bentuk Jacobi

Jika kita ambil x = sin ϕ, pada bentuk Legendre, maka akan didapat integral eliptik bentuk Jacobi jenis pertama dan kedua, sbb :

$$x = \sin\phi,$$

$$dx = \cos\phi \, d\phi \qquad \text{or} \qquad d\phi = \frac{dx}{\cos\phi} = \frac{dx}{\sqrt{1-x^2}},$$

$\phi = \pi/2$ corresponds to $x = 1$.

dan

$$F(k,\ \phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \int_0^{x} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}},$$

$$E(k,\ \phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \phi}\ d\phi = \int_0^{x} \sqrt{\frac{1 - k^2 x^2}{1 - x^2}}\ dx,$$

$$K = F\left(k,\ \frac{\pi}{2}\right) = \int_0^{1} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}},$$

$$E = \int_0^{1} \sqrt{\frac{1 - k^2 x^2}{1 - x^2}}\ dx.$$

Contoh soal

$$\int_0^{0,8} \frac{dx}{\sqrt{\left(1-x^2\right)\left(1-0,16x^2\right)}}$$

Contoh soal

1. $$\int_{0}^{5\pi/4} \sqrt{1-0{,}037\sin^2\phi}\ d\phi$$

3. $$\int_{-7\pi/8}^{11\pi/4} \sqrt{1-0{,}64\sin^2\phi}\ d\phi$$

2. $$\int_{0}^{0{,}5} \sqrt{\frac{100-x^2}{1-x^2}}\ dx$$

4. $$\int_{-0{,}5}^{0{,}5} \frac{dx}{\sqrt{(1-x^2)(4-3x^2)}}$$