# ANALYTICAL STUDY OF CHAOTIC SOLUTION OF AN AUTOPARAMETRIC SYSTEM WITH PARAMETRIC EXCITATION

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Abstract. We analytically study the existance of chaotic dynamics on Autoparametric system with parametric excitation. The method of averaging is used to yield a set of autonomous equation of the approximation to the response of the system. We use a global perturbation method developed by Kovacic and Wiggins to analyze the parameter range for wich a Shilnikov type Homoclinic orbit exists. This orbit gives rise to a well-described chaotic dynamics.

## 1. Introduction

This paper contains a further analysis of system first presented in [1]. There we considered an autoparametric system where the Oscillator is excited parametrically:

$$
\begin{aligned}
x'' + k_1 x' + q_1^2 x + ap(\tau)x + f(x,y) &= 0 \\
y'' + k_2 y' + q_2^2 y + g(x,y) &= 0
\end{aligned}
\tag{1.1}
$$

where $f(x,y) = c_1xy^2 + d_1x^3$, $g(x,y) = d_2y^3 + c_2x^2y$, and $p(\tau) = \cos 2\tau$. The natural frequencies $q_1$ and $q_2$ are both close to 1, so there exist a 1:1 internal resonance as well as 1:2 resonance with the external excitation. The nonlinear terms can be chosen more general. However, an averaging procedure will be used to study (1.1) and the indicated terms are the only ones that give a contribution, therefore there is no loss of generality in the choice of nonlinearity.

In [1] we studied the behavior of a stable periodic solution $x(\tau)$ of $x''+k_1x'+q^2x+f(x,0) = 0$. Various types of bifurcation of this solution were analyzed. Also, numerical simulation suggested the existence of non-trivial solutions which were either periodic, quasi-periodic or chaotic.

The aim of this paper is to show the existence of these non-trivial solution in a more rigorous, analytical way. To this end we combine the analysis of a codimension 2 bifurcation with the application of a generalized Melnikov method to yield a full picture of the dynamics of (1.1). The results of this theoretical analysis, in particular concerning the existence of chaotic solutions, show a remarkable degree of agreement with the numerical results.

## 2. The averaged System in Action Angle Variables

Writing $q_1^2 = 1 + \varepsilon\sigma_1$, $q_2^2 = 1 + \varepsilon\sigma_2$, scaling $k_i = \varepsilon\tilde{k}_i$, $c_i = \varepsilon\tilde{c}_i$, $d_i = \varepsilon\tilde{d}_i, i = 1, 2$, $a = \varepsilon\tilde{a}$, and $t = \varepsilon\tau$. Transforming $x = u_1 \cos\tau + v_1 \sin\tau$, $y = u_2 \cos\tau + v_2 \sin\tau$, and performing

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an averaging procedure, then rescaling $\tau = \frac{\varepsilon}{2}\tilde{\tau}$, see [1] for details. In the sequel a different formulation of (2.1) will often be used transformation to action-angle variables

$$u_i = -\sqrt{2R_i}\cos\theta_i \quad \text{and} \quad v_i = \sqrt{2R_i}\sin\theta_i, \quad i = 1, 2 \tag{2.1}$$

yields

$$\begin{aligned} R_1' &= -2k_1R_1 + c_1R_1R_2\sin(2\theta_1 - 2\theta_2) + aR_1\sin 2\theta_1 \\ R_1\theta_1{}' &= \sigma_1R_1 + \frac{1}{2}aR_1\cos 2\theta_1 + 2R_1^2 + c_1R_1R_2 + \frac{1}{2}c_1R_1R_2\cos(2\theta_1 - 2\theta_2) \\ R_2' &= -2k_2R_2 - c_2R_1R_2\sin(2\theta_1 - 2\theta_2) \\ R_2\theta_2{}' &= \sigma_2R_2 + \frac{1}{2}c_2R_1R_2\cos(2\theta_1 - 2\theta_2) + 2R_2^2 + c_2R_1R_2 \end{aligned} \tag{2.2}$$

## 3. Analytical Study of Chaotic Solutions by Using a Generalized Melnikov Method

By using a generalized version of Melnikov's method, we show that for certain values of the parameters, the averaged system (2.3) has a homoclinic orbit of Shilnikov-type. As is well-known [2], the existence of a Shilnikov-orbit implies the existence of chaotic dynamics. The method used here is based on Kovačič and Wiggins. To apply the method, a rescaling of the parameters is needed which leads to a system where the unperturbed part is Hamiltonian and integrable within the 4-dimensional phase-space. The unperturbed system possesses a 2-dimensional invariant manifold $M$, with a 3-dimensional stable manifold. Both the invariant manifold and its stable manifold survive when the perturbations are added. Within the invariant manifold $M_\varepsilon$ we identify a fixed point $p_\varepsilon$. By using a Melnikov method, we can find the range of parameters for which the 1-dimensional unstable manifold of $p_\varepsilon$ intersects the stable manifold of $M_\varepsilon$. Under certain conditions, this yields a homoclinic orbit of Shilnikov-type. We present the details below.

3.1. **Transformation to Hamiltonian Coordinates.** We first introduce the following transformations:

$$\begin{aligned} &q_1 = 2\theta_1 - 2\theta_2, \quad q_2 = 2\theta_2, \quad p_1 = -c_2R_1, \\ &p_2 = p_1 - c_1R_2, \quad \text{where} \quad c_2 < 0 \quad \text{and} \quad c_1 > 0. \end{aligned} \tag{3.1}$$

Note that, because $R_2 \geq 0$, we are only interested in the area of phase-space where $P_2 \leq P_1$. In particular, the hyper-plane $P_2 = P_1$ corresponds to the invariant space $R_2 = 0$.

After transforming system (2.2) using (3.1) and then rescaling the variables and parameters by $p_{1,2} \to \varepsilon P_{1,2}$, $q_{1,2} \to -2Q_{1,2}$, $k_{1,2} \to \varepsilon^2\bar{k}_{1,2}$, $\sigma_{1,2} \to \varepsilon\bar{\sigma}_{1,2}$, $a \to \varepsilon^2\bar{a}$, and $\tau \to 2\varepsilon\bar{\tau}$, system (2.2) becomes

$$
\begin{aligned}
P_1' &= \frac{\partial H_o}{\partial Q_1} + \varepsilon\left(-4\bar{k_1}P_1 + \frac{\partial H_1}{\partial Q_1}\right)\\
Q_1' &= -\frac{\partial H_o}{\partial P_1} - \varepsilon\frac{\partial H_1}{\partial P_1}\\
P_2' &= \varepsilon\left(\frac{\partial H_1}{\partial Q_2} + 2\kappa P_1 - 4\bar{k_2}P_2\right)\\
Q_2' &= -\frac{\partial H_o}{\partial P_2}
\end{aligned}
\tag{3.2}
$$

where

$$
\begin{aligned}
H_o &= \sigma P_1 + 2\bar{\sigma_2}P_2 + \bar{c_1}P_1P_2 - \frac{1}{2}\bar{c_3}P_2^2 - P_1(P_2 - P_1)(\bar{c_2} + \cos 2Q_1)\\
H_1 &= \bar{a}P_1\cos 2(Q_1 + Q_2)
\end{aligned}
\tag{3.3}
$$

and $\sigma = 2\bar{\sigma_1} - 2\bar{\sigma_2}$, $\bar{c_1} = \frac{2}{c_1} - \frac{2}{c_2}$, $\bar{c_2} = 2 - \frac{2}{c_1} - \frac{2}{c_2}$, $\bar{c_3} = \frac{4}{c_1}$, and $\kappa = 2\bar{k_2} - 2\bar{k_1}$. It is clear that for $\bar{k_1} = \bar{k_2} = 0$, system (4.2) is in canonical form, with Hamiltonian $H = H_o + \varepsilon H$. Note that $\frac{\partial H}{\partial Q_2} = 0$.

3.2. **Analysis of the Unperturbed System.** In this subsection we study the dynamic of the unperturbed ($\varepsilon = 0$) system. It is given by

$$
\begin{aligned}
P_1' &= 2P_1(P_2 - P_1)\sin 2Q_1\\
Q_1' &= -\sigma - \bar{c_1}P_2 - (2P_1 - P_2)(\bar{c_2} + \cos 2Q_1)\\
P_2' &= 0\\
Q_2' &= -2\bar{\sigma_2} - \bar{c_1}P_1 + \bar{c_3}P_2 + (\bar{c_2} + \cos 2Q_1)P_1
\end{aligned}
\tag{3.4}
$$

System (3.4) is integrable, since it possesses the independent integrals $H_o$ and $P_2$. In fact, it can in principle be integrated because the equations for $P_1$ and $Q_1$ are decoupled from the other two equations. We will first study the equations for $P_1$ and $Q_1$, taking $P_2$ as a constant.

We are only interested in studying the dynamics of these equations in the range $0 < P_2 \leq P_1$ and $0 < Q_1 < \pi$, since the equations are $\pi$-periodic in $Q_1$. One set of fixed points is given by $P_1 = P_2$ and $Q_1$ a solution of

$$
\cos 2Q_{1s} = -\frac{1}{P_2}(\sigma + \bar{c_1}P_2 + \bar{c_2}P_2)
\tag{3.5}
$$

This yields solution $Q_{1s}$ and $\pi - Q_{1s}$, provided that $0 < P_2 < P_1$. A simple stability analysis shows that these points are of saddle type. Note that these points are connected through a heteroclinic orbit on the invariant line $P_1 = P_2$. We also note that this invariant line $P_1 = P_2$ corresponds, in the original coordinates, with the invariant space $R_2 = 0$, i.e. $y = 0$. Therefore, these two fixed points correspond to semi-trivial solutions, Another fixed point is given by $Q_1 = \frac{\pi}{2}$ and $P_1 = \bar{P_1} = \frac{-\sigma + (\bar{c_2} - \bar{c_1} - 1)P_2}{2(\bar{c_2} - 1)}$. From the condition that $P_1 \geq P_2$, it follows that such a $\bar{P_1}$ only exists for $P_{21} < P_2 < P_{22}$, with $P_{21} = -\sigma/(3 - \frac{4}{c_2})$ and $P_{22} = -\sigma/(1 - \frac{4}{c_2})$. This fixed point is a center-point, and in the original coordinates it represents a non-trivial periodic solution.

The orbits in the $(P_1, Q_1)$-plane are the level curves of the unperturbed Hamiltonian $H_o$ restricted to the plane. The orbits through the saddle points $(P_1, Q_1) = (P_2, Q_{1s})$ and

$(P_1, Q_1) = (P_2, \pi - Q_{1s})$ can be found by solving $H_\circ(P_1, Q_1) - H_\circ(P_2, Q_{1s}) = 0$ for $P_1$. We then have

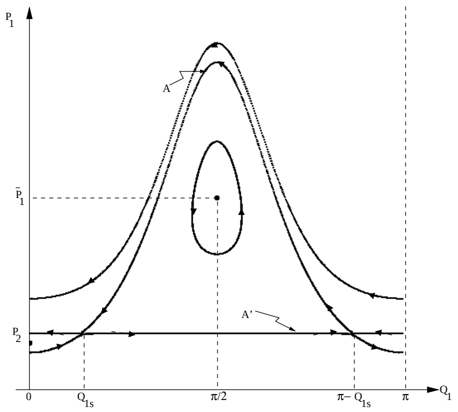


FIGURE 1. The phase-portrait of the unperturbed system in the $(P_1, Q_1)$-plane, for values $c_1 = 1$, $c_2 = -1$, $\bar{\sigma_1} = -8$, $\bar{\sigma_2} = 5.3$, and $P_2 = 4$.

$$\begin{aligned} &\text{orbit}A' : P_1 = P_2 \\ &\text{orbit}A : P_1 = -\frac{\sigma + \bar{c_1} P_2}{\bar{c_2} + \cos 2Q_1} \end{aligned} \tag{3.6}$$

These expressions for the heteroclinic orbits will be used later, when we apply the Melnikov method. The phase-portrait in the $(P_1, Q_1)$-plane is shown in Figure 1. Using Figure 1, we can get an impression of the dynamics of the unperturbed system in the full, four-dimensional, phase-space. Since the 2-dimensional phase-space for $P_1$ and $Q_1$ is qualitatively the same for all $P_{21} < P_2 < P_{22}$ and the equation for $Q_2$ is decoupled, we can picture the phase-space as in Figure 2.

The sets

$$\begin{aligned} M_1 &= \{(P_1, Q_1, P_2, Q_2) | P_1 = P_2, Q_1 = Q_{1s}, P_{21} < P_2 < P_{22}\} \\ M_2 &= \{(P_1, Q_1, P_2, Q_2) | P_1 = P_2, Q_1 = \pi - Q_{1s}, P_{21} < P_2 < P_{22}\} \end{aligned} \tag{3.7}$$

define a two dimensional invariant manifold $M = M_1 \cup M_2$ for the unperturbed equation. It is normally hyperbolic (see Wiggins[4]) which means that under linearized dynamics, the rates of expansion and contraction transverse to $M$ dominate those tangent to $M$.

The existence of the heteroclinic orbits joining $M_1$ and $M_2$ implies the nontranversal intersection of the three-dimensional stable manifold $W^s(M)$ and the three-dimensional unstable manifold $W^u(M)$ along a three-dimensional heteroclinic manifold $\Gamma$ (Figure 2), where

$$\begin{aligned} \Gamma &\equiv W^s(M) \cap W^u(M) \\ &= \{(P_1, Q_1, P_2, Q_2) | H_\circ(P_1, Q_1, P_2) - H_\circ(P_2, Q_{1s}, P_2) = 0\} \end{aligned} \tag{3.8}$$

The trajectories in $\Gamma$ approach a trajectory in $M$ asymptotically as $\tau \to \pm\infty$. By using the persistence theorem (see Fenichel [3]), we can show that $M$ persists under small perturbations as a locally invariant manifold $M_\epsilon$ with a boundary. Moreover, the manifolds $W^s(M)$ and $W^u(M)$ also persist as a locally invariant manifold $W^s(M_\epsilon)$ and $W^u(M_\epsilon)$.

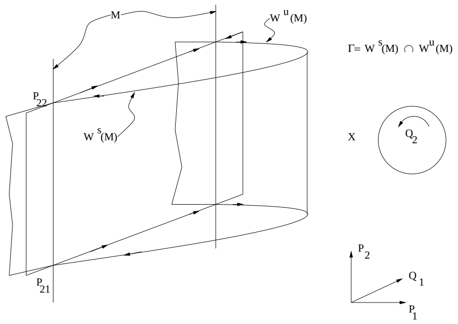


FIGURE 2. The Unperturbed system and manifolds $M$ in the $(P_1, Q_1, P_2)$-space.

The dynamics of the unperturbed system restricted to $M$ is given by

$$P_2' = 0$$
$$Q_2' = -2\bar{\sigma}_1 - \beta P_2 \tag{3.9}$$

See Figure 3a. Because the equations are $\pi$-periodic in $Q_2$, the phase-space is the cylinder obtained by identifying the edges $Q_2 = 0$ and $Q_2 = \Pi$. The phase-space therefore consists of a collection of invariant circles. However, $Q_2' = 0$ for $P_2{}^r = \frac{c_2\bar{\sigma}_1}{2}$. Therefore for $P_2{}^r = \frac{c_2\bar{\sigma}_1}{2}$ (the resonant value) we have a circle of fixed points.

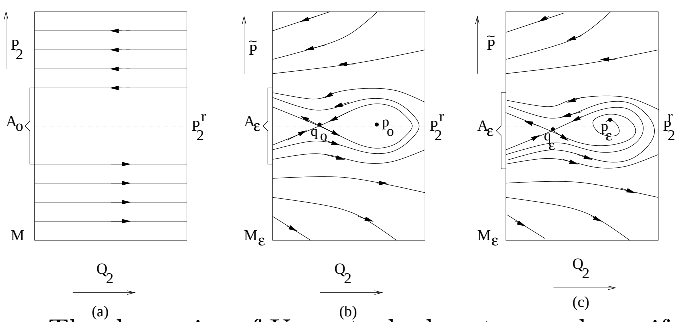


FIGURE 3. a. The dynamics of Unperturbed system and manifolds $A$ in the $(Q_2, P_2)$-plane. b. The perturbed system and manifolds $A_\varepsilon$ in the $(Q_2, P_2)$-plane.

To study the dynamics on the perturbed invariant manifold $M_\varepsilon$, we can use averaging over $Q_2$, as long as $P_2$ is not close to the resonant value $P_2{}^r = \frac{c_2\bar{\sigma}_1}{2}$. This yields the equation $\bar{P_2}' = \varepsilon(-4\bar{k_1})\bar{P}_2$, implying $\bar{P}_2 \to 0$ as $\bar{\tau} \to \infty$. This means that all these orbits eventually leave $M_\varepsilon$. However, the behavior near $P_2 = P_2{}^r$ is quite different.

To study the dynamics of the perturbed system restricted to $M_\varepsilon$ near the resonance $P_2 = P_2{}^r$ we will change variables.

Let $P_2 = {P_2}^r + \sqrt{\varepsilon}P$ and $\tau = \sqrt{\varepsilon}\tau$, we then have

$$
\begin{aligned}
P' =& -4\bar{k_1}{P_2}^r - 2\bar{a}{P_2}^r\sin 2(Q_{1s}+Q_2) + \sqrt{\varepsilon}(-4\bar{k_1}P - 2\bar{a}P\sin 2(Q_{1s}+Q_2)) \\
&+ O(\varepsilon) \\
Q_2' =& -\beta P + O(\varepsilon)
\end{aligned}
\tag{3.10}
$$

for $\varepsilon = 0$, the equation (3.10) is Hamiltonian, where the Hamiltonian is given by

$$
\bar{H} = -\frac{1}{2}\beta P^2 + \int 4\bar{k_1}{P_2}^r + 2\bar{a}{P_2}^r\sin 2(Q_{1s}+Q_2(\tau))d\tau \tag{3.11}
$$

The fixed points are $(P, Q_2) = (0, Q_{2c})$ and $(P, Q_2) = (0, Q_{2s})$. We will denote these fixed points as $p_\circ$ and $q_\circ$, respectively. In the range $0 < 2\bar{k_1} < \bar{a}$ and $0 < Q_{1s} < \pi$, we have $-\pi < Q_2 < -\frac{1}{4}\pi$. In this range $Q_{2c}$ and $Q_{2s}$ are represented by

$$
\begin{aligned}
Q_{2c} &= \frac{1}{2}\sin^{-1}\left[\frac{-2\bar{k_1}}{\bar{a}}\right] - Q_{1s} \quad \text{and} \\
Q_{2s} &= -\frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left[\frac{-2\bar{k_1}}{\bar{a}}\right] - Q_{1s}
\end{aligned}
\tag{3.12}
$$

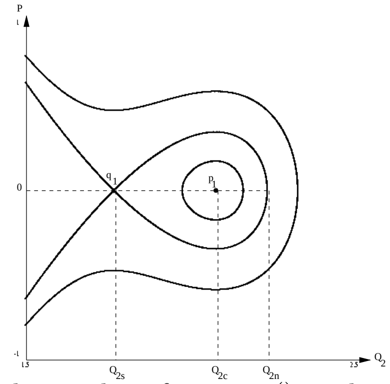


FIGURE 4. The homoclinic orbit of system () in the $(P, Q_2)$-plane, for values $\bar{k_1} = \bar{k_2} = 1$, $\bar{a} = 2.1$, $c_1 = 1$, $c_2 = -1$, $\bar{\sigma_1} = -8$, $\bar{\sigma_2} = 5.3$, and $P_2 = 4$.

The fixed point $p_\circ$ is a center point and the fixed point $q_\circ$ is a saddle point. The fixed point $q_\circ$ is connected to itself by a homoclinic orbit and $p_\circ$ is the only fixed point inside this homoclinic orbit. The range $Q_2$ in the homoclinic orbit is $Q_{2s} < Q_2 < Q_{2n}$ (see Figure 4), where $Q_{2n}$ can be solved from

$$
\bar{H}(0, Q_{2n}) - \bar{H}(0, Q_{2s}) = 0. \tag{3.13}
$$

The point of view of the dynamics on $M_\varepsilon$, we are only interested in the dynamics in an order $\sqrt{\varepsilon}$ near the resonance $P_2 = {P_2}^r$. To emphasizes this fact, we denote the annulus centered at $P_2 = {P_2}^r$ as $A_\varepsilon$. Since we will want to compare the dynamics in $A_\varepsilon$ with the unperturbed dynamics in the same region on $M_\varepsilon$, we define the unperturbed annulus as $A_\circ$. By restricting the $P_2$ values appropriately, we have the three dimensional stable and unstable manifolds of $A_\varepsilon$, denoted $W^s(A_\varepsilon)$ and $W^u(A_\varepsilon)$, respectively, are subset of $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$, respectively.

When the perturbation terms of order $\sqrt{\varepsilon}$ are taken into account, the perturbed $p_\varepsilon$ identical to $p_\circ$ and becomes sink due to $O(\sqrt{\varepsilon})$. Moreover, the homoclinic orbit breaks with a branch of unstable manifold of $q_\varepsilon$ falling into $p_\varepsilon$, see Figure 5 and 6. In Figure 6, the unperturbed system has one-dimensional $W^u(p_\circ)$ lying in the three dimensional manifold $W^s(A_\circ)$. By using the generalized Melnikov method, we can develop a measure of the distance between $W^u(p_\varepsilon)$ and $W^s(A_\varepsilon)$ and show that $W^u(p_\varepsilon) \subset W^s(A_\varepsilon)$ for the perturbed system shown in Figure 5.

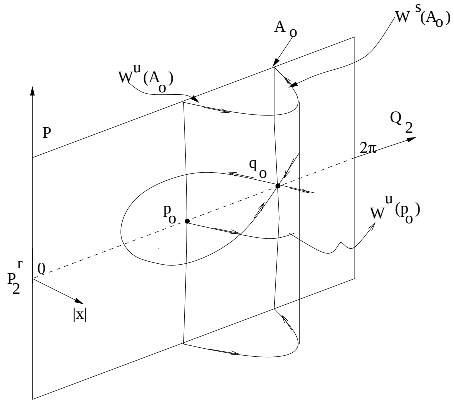


FIGURE 5. The dynamics in $A_\circ$ and its associated manifolds

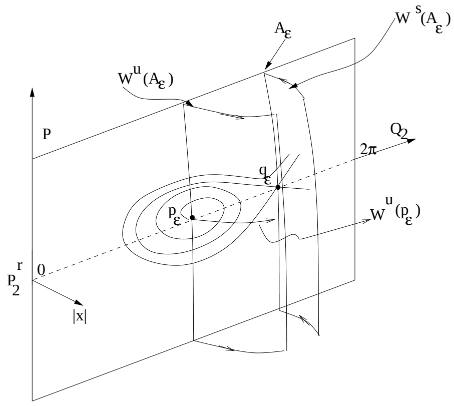


FIGURE 6. The dynamics in $A_\epsilon$ and its associated manifolds.

3.3. **Melnikov Function.** In calculating the Melnikov function it will be important to have forms for $P_1$, $Q_1$ and $Q_2$ as functions of time $\tau$. We substitute equation (3.10) into equation (3.5) and integrate. For orbit $A$, $Q_1(\tau)$ can implicitly be written as

$$\tanh(e_A\tau) = \frac{\sin 2Q_{1s} \sin 2Q_1(\tau)}{1 - \cos 2Q_{1s} \cos 2Q_1(\tau)} \tag{3.14}$$

where $e_A = -P_2 \sin 2Q_{1s}\mathrm{sgn}(\sin 2Q_{1s})$, and the expressions for $\cos 2Q_1(\tau)$ is

$$\cos 2Q_1(\tau) = \frac{\cos 2Q_{1s} \cosh(e_A\tau) - 1}{\cosh(e_A\tau) - \cos 2Q_{1s}} \tag{3.15}$$

Substituting equation (3.6) into (3.10), we have the explicit form for $P_1$ as function of time $\tau$. The expression is

$$P_1(\tau) = P_2 \frac{\cosh(e_A\tau) - \cos 2Q_{1s}}{\cosh(e_A\tau) + \cos(f_A)} \tag{3.16}$$

where

$$\cos(f_A) = -\frac{1 + \bar{c_2}\cos 2Q_{1s}}{\bar{c_2} + \cos 2Q_{1s}} \tag{3.17}$$

Finally, to calculate $Q_2$ as function of time $\tau$, we substitute equation (3.10) into (3.7), yield

$$Q_2' = c_A P_2 + \bar{c_1}(P_2 - P_1) \tag{3.18}$$

where $c_A = -\frac{2\bar{\sigma_1}}{P_2} + \bar{c_3} - 2\bar{c_1}$. After substituting equation (3.16) and (3.17) into (3.18), we thus have

$$Q_2' = c_A P_2 - \bar{c_1} P_2 \frac{\cos 2Q_{1s} + \cos(f_A)}{\cosh(e_A\tau) + \cos(f_A)} \tag{3.19}$$

On integrating equation (3.19) obtains

$$Q_2(\tau) = c_A P_2 \tau - g_A \tan^{-1}\left[\tan\left(\frac{f_A}{2}\right)\tanh\left(\frac{e_A\tau}{2}\right)\right] \tag{3.20}$$

where

$$g_A = 2\bar{c_1}\frac{\cos(f_A) + \cos 2Q_{1s}}{\sin 2Q_{1s}\sin(f_A)}\mathrm{sgn}(2Q_{1s}) \tag{3.21}$$

note that from equation (4.10), $P_1(\tau)$ a constant for orbit $A'$.

By letting $P_2 = {P_2}^r$, we compute the phase shift $\Delta Q_2$ of orbits which are asymptotic to points on the circle of fixed points as $\tau \to \pm\infty$. From equation (3.20), we have

$$\Delta Q_2 = Q_2(+\infty) - Q_2(-\infty) = g_A f_A \mathrm{sgn}(2Q_{1s}) \tag{3.22}$$

We now consider system (3.2). By taking $-Q_2$ instead of $Q_2$ as the angle conjugate to $P_2$ or by interchanging $P_1$ and $P_2$ and taking the negative of the Hamiltonian, we can put system (3.2) in the form of $(1.1)_\varepsilon$ in Wiggins [4]. Then we have the integrand of the Melnikov function as

$$\begin{aligned} &\frac{\partial H_\circ}{\partial P_1}\frac{\partial H_1}{\partial Q_1} - \frac{\partial H_\circ}{\partial Q_1}\frac{\partial H_1}{\partial P_1} - 4\bar{k_1}P_1\frac{\partial H_\circ}{\partial P_1} + \\ &\quad \left(\frac{\partial H_\circ}{\partial P_2}(P_1, Q_1, P_2) - \frac{\partial H_\circ}{\partial P_2}(P_1, Q_{1s}, P_2)\right)\left(-\frac{\partial H_1}{\partial Q_2} + 2\kappa P_1 - 4\bar{k_2}P_2\right)\end{aligned} \tag{3.23}$$

This Melnikov function integrand can be simplified by using the chain rule gives

$$\frac{\partial H_\circ}{\partial P_1}\frac{\partial H_1}{\partial Q_1} - \frac{\partial H_\circ}{\partial Q_1}\frac{\partial H_1}{\partial P_1} = -\frac{dH_1}{d\tau} - \frac{\partial H_1}{\partial Q_2}\frac{\partial H_\circ}{\partial P_2} \tag{3.24}$$

where we have used the fact that for $\varepsilon = 0$ then $P_2' = 0$, $\frac{\partial H_\circ}{\partial P_1} = -Q_1'$, $\frac{\partial H_\circ}{\partial P_2} = -Q_2'$, and evaluated the integrand at $P_2 = {P_2}^r$, the Melnikov function thus simplifies to

$$-\frac{dH_1}{dt} + 4\bar{k_1}P_1Q_1' + (-2\kappa P_1 + 4\bar{k_2}{P_2}^r - 4\bar{a}P_1\sin 2(Q_1 + Q_2))Q_2' \tag{3.25}$$

We now integrate (3.25) around the unperturbed heteroclinic orbit at $P_2 = {P_2}^r$ that approaches $p_\circ$ asymptotically as $\tau \to -\infty$.

It is clear that the first term in (3.25) can be integrated to give

$$-\int_{-\infty}^{+\infty}\frac{dH_1}{d\tau} = -\bar{a}P_2{}^r cos2(Q_1(\tau)+Q_2(\tau))|_{-\infty}^{+\infty} \tag{3.26}$$

The Melnikov function is evaluated on the orbit emanating from the center fixed point $p_\circ$, at the resonance value ($P = 0$). Recalling that $\Delta Q_2 = Q_2(+\infty) - Q_2(-\infty)$, and using $Q_1(\pm\infty) = \pm\mathrm{sgn}(e_A)$, $Q_2(-\infty) = p_\circ$,
$Q_2(+\infty) = \Delta Q_2 + p_\circ$, and trigonometric identities allow us to simplify (3.26) to

$$\begin{aligned}-\int_{-\infty}^{+\infty}\frac{dH_1}{d\tau} =& -\bar{a}P_2{}^r(\cos 2p_\circ[\cos 2Q_{1s}\,(\cos 2\Delta Q_2 - 1)\\ &- \mathrm{sgn}(e_A)\sin 2Q_{1s}\sin 2\Delta Q_2] - \sin 2p_\circ[\cos 2Q_{1s}\sin 2\Delta Q_2\\ &+ \mathrm{sgn}(e_A)\sin 2Q_{1s}])\end{aligned} \tag{3.27}$$

The second term in (3.25) can be integrated by using the relation in equation (3.10) to obtain

$$\int_{-\infty}^{+\infty} 4\bar{k_1}P_1Q_1'd\tau = -8\bar{k_1}(\sigma+\bar{c_1}P_2{}^r)\frac{\tan^{-1}(\sqrt{\bar{c_2}^2-1})}{\sqrt{\bar{c_2}^2-1}}\tan Q_{1s}\mathrm{sgn}(e_A) \tag{3.28}$$

and the third term in (3.25) can be integrated as

$$\begin{aligned}&\int_{-\infty}^{+\infty} -2\kappa P_1 + 4\bar{k_2}P_2{}^r - 4\bar{a}P_1\sin 2(Q_1+Q_2)Q_2'd\tau =\\ &4\bar{k_2}P_2{}^r\Delta Q_2 + 2\kappa(\sigma + c_1\bar{P_2}^r)\int\frac{Q_2'd\tau}{\bar{c_2}+\cos 2Q_1(\tau)}\\ &-4\bar{a}(\sigma+\bar{c_1}P_2{}^r)\int\frac{\sin 2(Q_1(\tau)+Q_2(\tau))}{\bar{c_2}+\cos 2Q_1(\tau)}Q_2'd\tau\end{aligned} \tag{3.29}$$

From equations (3.24)- (3.29) the Melnikov function can be wrote down as

$$M(\mu) = M_1(\mu) + \sin Q_{20}(\sigma+\bar{c_1}P_2{}^r)M_2(\mu) \tag{3.30}$$

where $\mu = (\bar{\sigma_1}, \bar{\sigma_2}, \bar{a}, Q_{20})$.

Solving the Melnikov function $M(\mu) = 0$, we find the condition when a Silnikov type heteroclinic cycle exists. Such a cycle implies the existence of chaotic dynamics.

## 4. Conclusion

We have studied global bifurcations an autoparametric system of the form (1.1) with the conditions stated in [1]. There is explicit results of study of global bifurcation of the non-trivial solution. We find the Melnikov function indicating that the heteroclinic connection which exists , is not broken by perturbation. We also find the condition when a Silnikov type heteroclinic cycle exists. That cycle implies the existence of chaotic dynamics.

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