PARAMETRIC EXCITATION IN SELF-EXCITED THREE-DEGREES OF FREEDOM PROBLEMS

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In model 1 [1], we consider a three-mass system consisting of a central mass and two external masses . The ends of the external masses are elastically mounted by using springs of variable stiffness. The scheme of this model is shown in Figure 1. The central mass m and the external masses m_1 and m_2 represent reduced concentrated masses of body elements while the connecting springs simulate their elasticity. The springs have constant stiffness k. The linear viscous damping of the external masses m_1 and m_2 have coefficient b_o, while the variable stiffness is periodically varying in time as $k_o(1+\varepsilon\cos\omega t)$. The central mass m is self-excited by flow with a negative linear damping coefficient $-b$ and damped by the non-linear speed-dependent damping with coefficient c. The deflections from equilibrium positions are y_j $(j = 1, 2, 3)$.

In model 2, the system also consists of a central mass and two external masses, see Figure 2. The coordinates of masses $m_i,\ i = 1, 2, 3$ are denoted by y_i. The central mass and the external masses are connected by springs with the same constant stiffness k. The flow-generated self-excited force is acting on the external masses m_1 and m_2; it is represented by Rayleigh force in the form $bU(1 - \gamma_o\dot{y}_i^2)\dot{y}_i,\ i = 1, 3$, where b and γ_o are positive coefficients and U is the flow velocity. The linear viscous damping of the central mass m has coefficient b_o, while the variable stiffness is periodically varying in time as $k_o(1 + \varepsilon\cos\omega t)$.

In Figure 3, we consider a three mass system of model 3 where one of the ends of the external masses is mounted by using a spring of variable stiffness. A flow induced-vibration is acting on the external mass m_1 and the central mass m with the negative linear damping $-b_1$ and $-b_2$, respectively. The connecting springs have the same constant stiffness k. The external mass m_2 is supported by a spring with constant stiffness k_o and a linear viscous damper with damping parameter b_o.

The considered system of model 1 is governed by the following differential equations:

$$
\begin{aligned}
m_1\ddot{y}_1 + b_o\dot{y}_1 + k_o(1+\varepsilon\cos\omega t)y_1 + k(y_1 - y_2) &= 0\\
m\ddot{y}_2 - bU^2(1 - c\dot{y}_2^2)\dot{y}_2 + 2ky_2 - k(y_1 + y_3) &= 0\\
m_2\ddot{y}_3 + b_o\dot{y}_3 + k_o(1+\varepsilon\cos\omega t)y_3 + k(y_3 - y_2) &= 0
\end{aligned}
\tag{1}
$$

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The system of model 2 is governed by the following differential equations:

$$
\begin{aligned}
m_1\ddot{y}_1 + k(2y_1 - y_2) - bU^2(1 - \gamma_o\dot{y}_1^2)\dot{y}_1 &= 0 \\
m\ddot{y}_2 + k(2y_2 - (y_1 + y_3)) + k_o(1 + \varepsilon\cos\omega t)y_2 + b_o\dot{y}_2 &= 0 \\
m_2\ddot{y}_3 + k(2y_3 - y_2) - bU^2(1 - \gamma_\circ\dot{y}_3^2)\dot{y}_3 &=
\end{aligned} \tag{2}
$$

The system of model 3 is governed by the following differential equations:

$$
\begin{aligned}
m_1\ddot{y}_1 + k(y_1 - y_2) - b_1(1 - \gamma_o\dot{y}_1^2)\dot{y}_1 &= 0 \\
m_2\ddot{y}_2 + k(y_1 - y_2) + k_2(y_2 + y_3) - b_2(1 - \gamma_o\dot{y}_2^2)\dot{y}_2 + b_o\dot{y}_2 &= 0 \\
m_3\ddot{y}_3 - k(y_2 - y_3) + k_o(1 + \varepsilon\cos\omega t)y_3 &= 0
\end{aligned} \tag{3}
$$

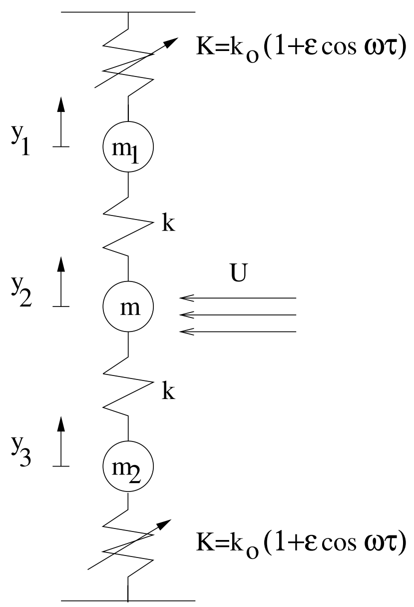


FIGURE 1. Model 1, the schematic representation of the three-mass chain system.

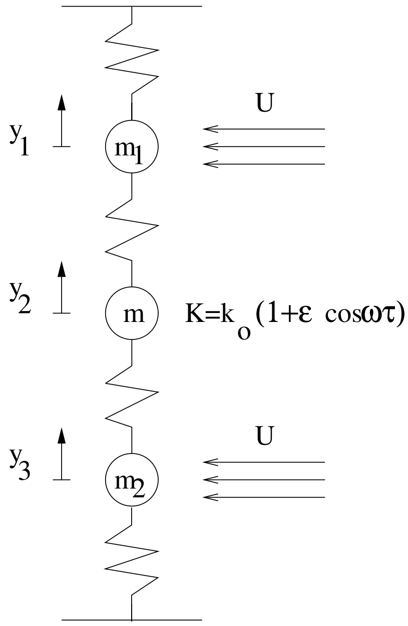


FIGURE 2. Model 2, the schematic representation of the three-mass chain system.

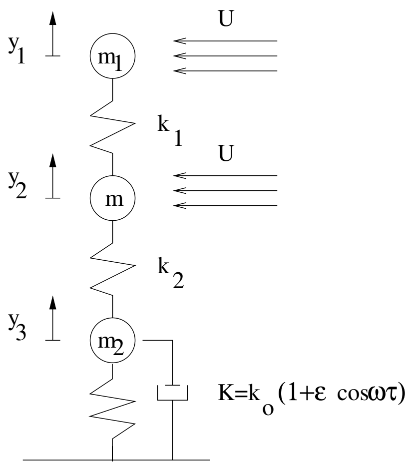


FIGURE 3. Model 3, the schematic representation of the three-mass chain system.

1. Analysis of Model 1

Tondl and Nabergoj [2] have been studied the case when $m_1 = m_2 = m_o$. Using the time transformation $\tau = \omega_o t$, where $\omega_o = \sqrt{2k/m}$ and the linear transformation

$$
\begin{aligned}
y_1 &= x_1 + x_2 + x_3 \\
y_2 &= a_1 x_1 + a_2 x_2 \\
y_3 &= x_1 + x_2 - x_3
\end{aligned} \tag{4}
$$

the system (1) can be transformed to two coupled quasi-normal Mathieu equations and one uncoupled Mathieu equation in the form

$$
\begin{aligned}
x_1'' + \Omega_1^2 x_1 + \varepsilon f_1(\mu_1, \cos\eta\tau, x_1, x_1', x_2, x_2', x_3, x_3') &= 0 \\
x_2'' + \Omega_2^2 x_2 + \varepsilon f_2(\mu_2, \cos\eta\tau, x_1, x_1', x_2, x_2', x_3, x_3') &= 0 \\
x_3'' + \Omega_3^2 x_3 + \varepsilon f_3(\cos\eta\tau, x_3, x_3') &= 0
\end{aligned} \tag{5}
$$

where the normal-mode frequencies depend on the ratio between the masses m_o and m. The parameters μ_1, μ_2 depend on θ_{11}, Q_{12} and θ_{22}, Q_{21}, respectively, where θ_{11}, θ_{22} are the damping coefficients, and Q_{12}, Q_{21} are the coefficients of the parametric term. The analysis of the linear case of system (5) shows that there are two conditions in order to obtain an interval of the frequency of parametric excitation η, where the trivial solution is stable. These conditions are found when the combination resonance $\eta = \Omega_2 - \Omega_1$ is considered.

The first condition is

$$\theta_{11} + \theta_{22} > 0. \tag{6}$$

The second condition for the stability interval boundaries is

$$\eta_o - \sigma < \eta < \eta_o + \sigma, \ \eta_o = \Omega_2 - \Omega_1 \tag{7}$$

where

$$\sigma = \varepsilon \frac{\theta_{11} + \theta_{22}}{\sqrt{|\theta_{11}\theta_{22}|}} \sqrt{-\frac{Q_{12}Q_{21}}{16\Omega_1\Omega_2} - \theta_{11}\theta_{22}} \tag{8}$$

Inside the interval the trivial solution is stable and it is unstable outside. For a more detailed study see Tondl and Nabergoj [2].

2. Analysis of Model 2

We study system (2) for the case $m_1 = m_2 = m$. Using the time transformation $\tau = \omega_o t$ with $\omega_o = \sqrt{2k/m}$, system (2) becomes

$$
\begin{aligned}
y_1'' + y_1 - \frac{1}{2}y_2 - \varepsilon\beta V^2(1 - \gamma {y_1'}^2)y_1' &= 0 \\
y_2'' + y_2 - \frac{1}{2}(y_1 + y_3) + q^2(1 + \varepsilon\cos\eta\tau)y_2 + \varepsilon\mu y_2' &= 0 \\
y_3'' + y_3 - \frac{1}{2}y_2 - \varepsilon\beta V^2(1 - \gamma {y_3'}^2)y_3' &= 0
\end{aligned} \tag{9}
$$

where

$$
\varepsilon\beta = \frac{b/m}{\omega_o},\ \eta = \frac{\omega}{\omega_o},\ q^2 = \frac{k_o/m}{\omega_o^2}U^2,\ \varepsilon\mu = \frac{b_o/m}{\omega_o},\ \gamma = \gamma_o\omega_o^2,\ \text{and } V = \frac{U}{U_o} \tag{10}
$$

System (9) can be transformed into a standard form (11) using the linear transformation (4).

$$
\begin{aligned}
x_1'' + \Omega_1^2 x_1 + \varepsilon f_1(\alpha_1, \cos\eta\tau, \mathbf{x}) &= 0 \\
x_2'' + \Omega_2^2 x_2 + \varepsilon f_2(\alpha_2, \cos\eta\tau, \mathbf{x}) &= 0 \\
x_3'' + x_3 + \varepsilon f_3(\theta_{31}, \mathbf{x}) &= 0
\end{aligned} \tag{11}
$$

where

$$
\mathbf{x} = (x_i,\ i = 1, 2, 3),\ \alpha_i = (\theta_{ij}, Q_{ij};\ i = 1, 2,\ j = 1, 2).
$$

The $x_i,\ i = 1, 2, 3$ are the normal coordinates corresponding to free vibrations of the system. The normal-mode frequencies $\Omega_{1,2}$ and the constant multipliers $a_{1,2}$ are given by the relations:

$$
\begin{aligned}
\Omega_{1,2} &= \frac{1}{2}(q^2 + 2) \mp \frac{1}{2}\sqrt{q^4 + 2}, \\
a_{1,2} &= -q^2 \pm \sqrt{q^4 + 2},
\end{aligned} \tag{12}
$$

where $q \neq 0$ and $q \neq 1$. We note that $\Omega_2 > \Omega_1 > 0$, $a_1 > 0$, $a_2 < 0$ and

$$
\begin{aligned}
f_i &= \frac{1}{2(a_1 - a_2)}(\theta_{i1}x_1' + \theta_{i2}x_2' + (Q_{i1}x_1 + Q_{i2}x_2)\cos\eta\tau \\
&\quad \mp 2a_{2,1}\beta V^2\gamma(3(x_1' + x_2')^2 x'_3 + (x_1' + x_2')^3)),\ i = 1, 2. \\
f_3 &= \frac{1}{2(a_1 - a_2)}(\theta_{31}x_3' + 4(a_1 - a_2)\beta V^2\gamma({x_3'}^3 + x_3'(x_1' + x_2')^2)
\end{aligned} \tag{13}
$$

We note that system (13) is an Autoparametric system where $x_3 = 0$ corresponds to the semi-trivial solution of the system. Next, we use the parameters θ_{11}, θ_{22}, Q_{12}, and Q_{21}, where

$$
\begin{aligned}
&\theta_{11} = 2(a_1\mu + a_2\beta V^2),\ \theta_{22} = -2(a_2\mu + a_1\beta V^2) \\
&\theta_{31} = -2(a_1 - a_2)\beta V^2,\ Q_{12} = 2q^2 a_2,\ Q_{21} = -2q^2 a_1
\end{aligned} \tag{14}
$$

2.1. **The Normal Form by Averaging.** We note that the parametric excitation terms only occur in the normal coordinates x_1, x_2. There are three natural frequencies of system (11), i.e, Ω_1, Ω_2, and $\Omega_3 = 1$. Due to occurrence of parametric resonance and self-excitation of system (11), we consider the external resonance $\eta = \Omega_2 - \Omega_1$ and the internal resonance $\Omega_2 - \Omega_3 - 2\Omega_1 = 0$. Transforming $t \to \eta\tau$ and allowing detuning near $\eta_\circ$ by putting

$$\eta = \eta_o + \varepsilon\bar{\sigma}, \;\; \eta_o = \Omega_2 - \Omega_1 \tag{15}$$

we then transform system (11) by using Lagrange transformation,

$$\begin{aligned} x_i &= u_i \cos\omega_i t + v_i \sin\omega_i t, \\ \dot{x}_1 &= -\omega_i u_i \sin\omega_i t + \omega_i v_i \cos\omega_i t, \end{aligned} \tag{16}$$

for $i = 1, 2, 3$ and $\omega_i = \frac{\Omega_i}{\eta}$. We use again the dot to indicate derivation with respect to the re-scaled time. After averaging over 2π and then rescaling time through $\frac{\varepsilon}{2(a_1-a_2)\eta_\circ^2}$, the first order in ε of the averaged system is of the form;

$$\dot{\mathbf{U}} = A\mathbf{U} + \mathbf{F}(\mathbf{U}) \tag{17}$$

where $\mathbf{U}$ is a vector $(u_i, v_i, \; i = 1, 2, 3)$ and $\mathbf{F}$ is a vector function $(f_i, \; i = 1..6)$. The function $\mathbf{F}$ only contains a cubic nonlinearity. The constant 6×6-matrix A is in the form

$$A = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix} \tag{18}$$

where $A_{ij}, \; i, j = 1, 2, \; \emptyset$ and A_{33} are 2×2-matrix. System (17) can be reduced to the five-dimension system by transforming the system using the following transformation,

$$u_i = -R_i \cos\psi_i, \quad \text{and} \quad v_i = R_i \sin\psi_i, \quad i = 1, 2, 3. \tag{19}$$

This transformation is useful for studying the semitrivial solution $(x_1, x_2, 0)$ of system (12) when $x_1 \neq 0, \; x_2 \neq 0$.

2.2. **The Semitrivial solution.** Consider $\lambda_i, \; i = 1..6$ which are the eigenvalues of matrix A. We find that the real parts of λ_5 and λ_6 of the trivial solution are positive. Then, the trivial solution of system (17) is always unstable. Let $(x_{10}, x_{20}, 0)$ be a semitrivial solution of system (17), where $x_{10}, \; x_{20}$ correspond with the non-trivial solutions $R_{10}, \; R_{20}$, and Ψ_0 of the following system

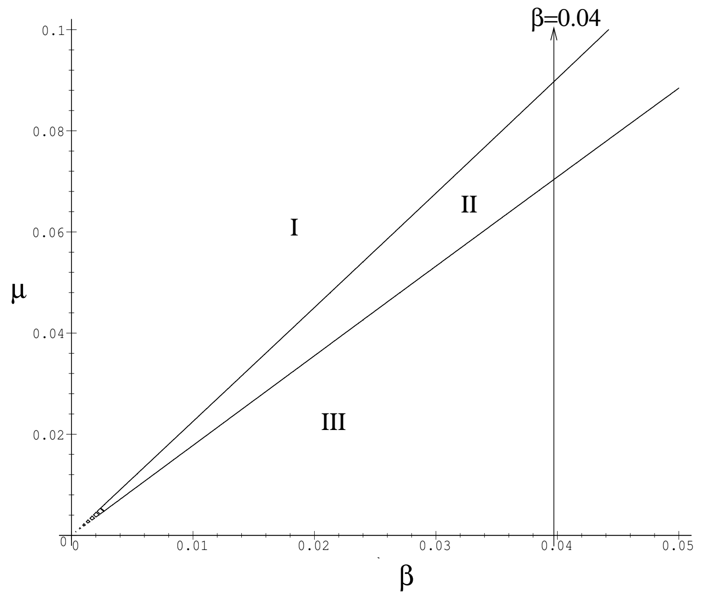


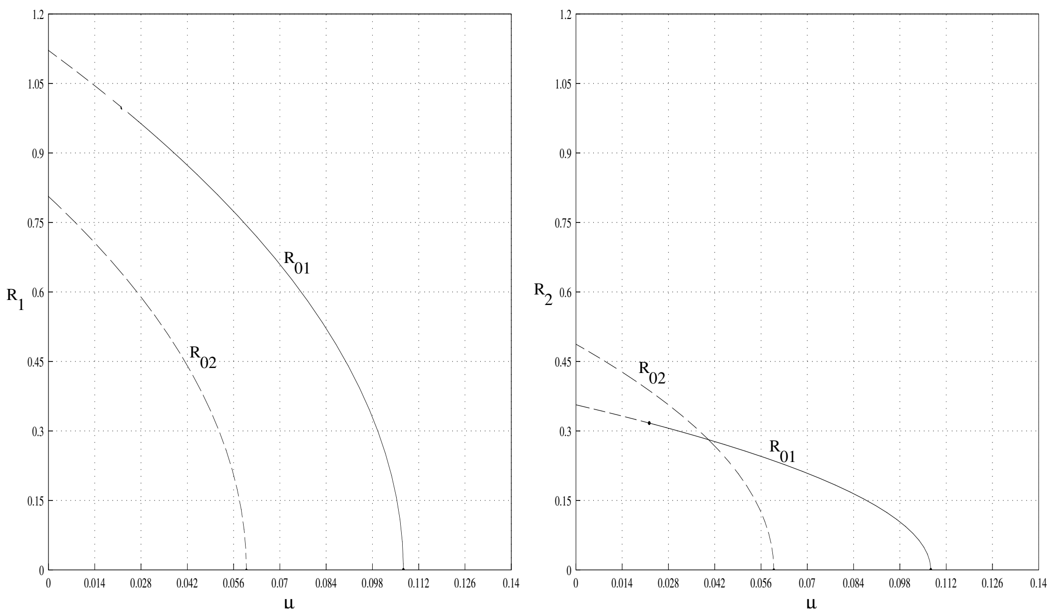
FIGURE 4. The parameter diagram in the (β, μ)-plane for fixed $q = 0.85$, $V = \sqrt{2}$, $\gamma = 1$, and $\varepsilon = 0.1$ for nontrivial fixed points of system (20). We find that there is no fixed point in region I. One fixed point $\mathbf{R_{01}}$ exists in region II. There are two fixed points $\mathbf{R_{01}}$ and $\mathbf{R_{02}}$ in region III. The line $\beta = 0.04$ is used in the numerical example in Figure 5.

$$\begin{aligned}
\dot{R}_1 &= -\theta_{11}\eta_o R_1 + \frac{1}{2}\frac{Q_{12}}{\omega_1}R_2\sin(\Psi) + \frac{3}{2}a_2\alpha\eta_o^3 R_1(\frac{1}{2}\omega_1^2R_1^2 + \omega_2^2R_2^2) \\
\dot{R}_2 &= -\theta_{22}\eta_\circ R_2 - \frac{1}{2}\frac{Q_{21}}{\omega_2}R_1\sin\Psi - \frac{3}{2}a_1\alpha\eta_o^3 R_2(\omega_1^2R_1^2 + \frac{1}{2}\omega_2^2R_2^2) \\
\dot{\Psi} &= 2(a_1 - a_2)\eta_\circ\bar{\sigma} + \frac{1}{2}(\frac{Q_{21}}{\omega_2}\frac{R_1}{R_2} - \frac{Q_{12}}{\omega_1}\frac{R_2}{R_1})\cos\Psi.
\end{aligned} \tag{20}$$

When we take $\dot{R}_1 = 0$, $\dot{R}_2 = 0$, and $\dot{\Psi} = 0$, we obtain fixed points of system (20). They correspond with periodic solutions of system (17).

The fixed points of system (20) are obtained by intersecting z_1 and z_2, where $z_1 \cap z_2 = \emptyset$ for $\bar{\sigma}_2 < \bar{\sigma} < \bar{\sigma}_1$ and $\bar{\sigma}_2 < 0$ and $\bar{\sigma}_1 > 0$. The explicit expression for $\bar{\sigma}_i$ $(i = 1, 2)$ can be found from equation (**??**).

Figure 5 shows the existence of the fixed point $\mathbf{R_0}$ when the parameter μ is varied along line $\beta = 0.04$. There is no fixed point of system (20) for $\mu > 0.10736$. The fixed point $\mathbf{R_{01}^+}$ exists in the interval $0.05987 < \mu < 0.10736$. There are two fixed points $\mathbf{R_{01}^+}$ and $\mathbf{R_{02}^-}$ in the interval $0.02219 < \mu < 0.05987$. Two fixed points $\mathbf{R_{01}^-}$ and $\mathbf{R_{02}}^-$ exist for $0 < \mu < 0.02219$. The $\mathbf{R_0^+}$ and $\mathbf{R_0^-}$ show that the fixed point $\mathbf{R_0}$ is attracting and it is non attracting, respectively,



(a) (b)

FIGURE 5. The existence of the fixed point $\mathbf{R_0}$ of system (17), for fixed $q = 0.85$, $V = \sqrt{2}$, $\gamma = 1$, $\varepsilon = 0.1$, and $\beta = 0.04$, (a) in the (μ, R_1)-plane and (b) in the (μ, R_2)-plane. There is no fixed point for $\mu > 0.10736$. There is one solution $\mathbf{R_{01}^+}$ for $0.05987 < \mu < 0.10736$. There are two solutions $\mathbf{R_{01}^+}$ and $\mathbf{R_{02}^-}$ in the interval $0.02219 < \mu < 0.05987$. Two solutions $\mathbf{R_{01}^-}$ and $\mathbf{R_{02}}^-$ exist for $0 < \mu < 0.02219$. The solid curve shows that the solution $\mathbf{R_0}$ is attracting in the (R_1, R_2)-plane. The dashed curve shows that it is non attracting.

in the (R_1, R_2)-plane. The solid curve shows that the fixed point $\mathbf{R_0}$ is attracting in the (R_1, R_2)-plane. The dashed curve shows that it is non attracting. We note that the fixed point $(\mathbf{R_0}, \mathbf{0})$ is always unstable in the full system.

In a further study one has to analyze the behavior of this unstable semitrivial solution $(\mathbf{R_0}, \mathbf{0})$ in the full system.