CHARACTERIZATION OF MAXIMAL PRIMITIVE IDEALS OF TOEPLITZ ALGEBRAS

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Abstract. The upwards-looking topology which was introduced by Adji and Raeburn corresponds to the hull-kernel topology in the primitive ideal space $\operatorname{Prim} \mathcal{T}(\Gamma)$ of Toeplitz algebra $\mathcal{T}(\Gamma)$ of totally ordered abelian group Γ. In this paper we discuss the closed sets in $\operatorname{Prim} \mathcal{T}(\Gamma)$ with the upwards-looking topology and characterize maximal primitive ideals.

1. Introduction

In [2] Adji and Raeburn studied the ideal structure of the Toeplitz algebras of a totally ordered abelian groups. Let Γ be a discrete totally ordered abelian group. An order ideal of a totally ordered abelian group Γ is a subgroup I which is order preserving, in the sense that if $x \in \Gamma^+, y \in I^+$ with $x \leq y$ then $x \in I$. The set $\Sigma(\Gamma)$ of order ideals is totally ordered by inclusion. If I is an order ideal of Γ , (I, γ) denotes a character $\gamma \in \hat{I}$, so if the order ideals I, J are not coincide, we note that $(J, \gamma) \neq (I, \gamma)$.

It was shown in [2] that each irreducible representation of the Toeplitz algebra $\mathcal{T}(\Gamma)$ factors through an irreducible representation of $\mathcal{T}(\Gamma/I)$ for some $I \in \Sigma(\Gamma)$, and that there is a bijection map L of the disjoint union

$$X(\Gamma) := \bigsqcup \{\hat{I} : I \in \Sigma(\Gamma)\} = \{(J, \gamma) : J \in \Sigma(\Gamma)\}$$

onto the set $\operatorname{Prim} \mathcal{T}(\Gamma)$ of primitive ideals of $\mathcal{T}(\Gamma)$.

Using the bijection L from $X(\Gamma)$ onto $\operatorname{Prim} \mathcal{T}(\Gamma)$, Adji and Raeburn describe a topology on $X(\Gamma)$ which corresponds to the hull-kernel topology on $\operatorname{Prim} \mathcal{T}(\Gamma)$. The topology on $X(\Gamma)$ is called the upwards-looking topology $X(\Gamma)$ (see Definition 1).

In this paper we firstly discuss some properties of the topology and the relationship between the point wise topology and the relative upwards-looking topology on $\hat{\Gamma}$. Secondly we characterize maximal primitive ideals of Toeplitz algebras which is given in Theorem 11.

2. The Upwards-Looking Topology

Throughout this paper, $(\Gamma, +)$ is a totally ordered discrete abelian group. Let I be an order ideal of Γ. The quotient group Γ/I is defined in the similar manner of usual quotient group, and there is a quotient order on Γ/I defined by

$$xI \leq yI \Leftrightarrow \exists i \in I \text{ such that } x \leq y + i,$$

this is also a total order. Moreover, if $q : \Gamma \longrightarrow \Gamma/I$ is the quotient homomorphism then $q(\Gamma^+) = (\Gamma/I)^+$.

Definition 1 (Upwards-Looking Topology). [2, Definition 4.1]
The closure $\bar{F}^{ULT}$ of a subset F of $X(\Gamma)$ consists of the pairs (J, γ) where J is an order ideal and $\gamma \in \hat{J}$ satisfies: for every open neighbourhood N of γ in $\hat{J}$, there exist $I \in \Sigma(\Gamma)$ and $\chi \in N$ such that $I \subset J$ and $(I, \chi|_I) \in F$.

It was verified in [2, Lemma 4.2] that the closure operation in the definition above satisfies the Kuratowski closure axiom, hence it generates a topology in $X(\Gamma)$. Thus the definition above describes a topology for $X(\Gamma)$. In this paper we are dealing with many topologies, we need to state our convention. We write $\bar{F}^{ULT}$ to denote the closure of a subset $F \subset X(\Gamma)$ in the upwards-looking topology for $X(\Gamma)$, and we write $\bar{F}^{PWT}$ to denote the closure of a subset $F \subset \hat{I}$ in the point wise topology for $\hat{I}$.

The next theorem gives the set of basis of open sets in the upwards-looking topology.

Theorem 2. [7, Lemma IV.18] *Suppose* Γ *is a totally ordered abelian group. For* $I \in \Sigma(\Gamma)$ *and open set* $M \subset \hat{I}$ *we define* $\mathcal{O}_{I,M} := \{(J, \gamma|_J) : J \subset I, \gamma \in M\}$. *Then*

(2.1) $\{\mathcal{O}_{I,M} : I \in \Sigma(\Gamma), M$ *is open in the point wise topology on* $\hat{I}\}$

is a set of basis of open sets for the topology upwards-looking topology described in Definition 1 .

Proof. To see $\mathcal{O}_{I,M}$ is open, we show that its complement is closed. For this, let $(K, \gamma) \in \overline{X(\Gamma) \setminus \mathcal{O}_{I,M}}$. We want $(K, \gamma) \in X(\Gamma) \setminus \mathcal{O}_{I,M}$, i.e $K \supsetneqq I$ or $\gamma \notin M$. Assume that $\gamma \in M$ and we claim that $K \supsetneqq I$. That $K = I$ is impossible. To see this, take M as an open neighbourhood of γ in $\hat{I}$ and there is $\chi \in M$, $J \in \Sigma(\Gamma)$ such that $(J, \chi|_J) \in X(\Gamma) \setminus \mathcal{O}_{I,M}$. But this implies that $J \supsetneqq I$ or $\chi \notin M$, which both give contradictions. That $K \subsetneqq I$ is also impossible. Because for any open neighbourhood N of γ in $\hat{K}$, there is $\chi \in N$, $J \in \Sigma(\Gamma)$ such that $(J, \chi|_J) \in X(\Gamma) \setminus \mathcal{O}_{I,M}$ which implies that $J \supsetneqq I$ or $\chi \notin M$, which again both of them give contradictions. Therefore $\mathcal{O}_{I,M}$ is open.

We next show that (2.1) is a basis of open sets for the topology described in Definition 1. Given $(J, \gamma) \in X(\Gamma)$ and a closed set F such that $(J, \gamma) \notin F$. Definition 1 implies that there is a neighbourhood N of γ in $\hat{J}$ such that $(I, \tau|_I) \notin F$ for all $\tau \in N$, $I \subset J$. So $(J, \gamma) \in \mathcal{O}_{J,N} \subset X(\Gamma) \setminus F$.   □

Example 3. We are going to discuss some description of sets in $X(\Gamma)$ by considering specific cases of Γ. An observation on $\Gamma := \mathbb{Z} \oplus_{\text{lex}} \mathbb{Z}$ gives interesting results. Let I be the ideal $\{(0, n) : n \in \mathbb{Z}\}$, since I is the only ideal, we have $X(\Gamma) = \hat{0} \sqcup \hat{I} \sqcup \hat{\Gamma}$. Suppose λ_0 is a character in $\hat{I}$ defined by $(0, n) \mapsto e^{2\pi i n}$, and let $F = \{\lambda_0\}$. Next we consider a character γ in $\hat{\Gamma}$ defined by $(m, n) \mapsto e^{2\pi i(m+n)}$. It is clear that $\gamma|_I = \lambda_0$. Then $\gamma \in \bar{F}^{ULT}$, because every open neighbourhood N of γ in $\hat{\Gamma}$ contains an element λ (which is nothing but γ it self) such that its restriction on I gives a character in F. It is clear that $\gamma \notin F$, hence F is not closed in the upwards-looking topology for $X(\Gamma)$.

The example above implies that any closed set in the point wise topology is not necessarily closed in the upwards-looking topology. Similarly, when we apply to any complement F^C of a set F, we arrive to a conclusion that any open set in the point wise topology, is not necessarily open in the upwards-looking topology. This observation gives a similar result for more general cases.

Proposition 4. *Let* Γ *be a totally ordered abelian group which has a nontrivial order ideal. Suppose* $I \in \Sigma(\Gamma)$ *such that* $I \neq \Gamma$. *Then any subset* $F \subset \hat{I}$ *is not closed. Moreover* $X(\Gamma)$ *is not a* T_1 *space.*

Proof. Let $\lambda \in F$, and let χ be an extension of λ to a character in $\hat{\Gamma}$. We are going to show that $\chi \in \bar{F}$, and hence F is not closed. Let N be any open neighbourhood of χ in $\hat{\Gamma}$. Since $\chi \in N$ and $\chi|_I = \lambda \in F$, the character χ is in the closure $\bar{F}^{ULT}$ of F. But $\chi \notin F$, hence F is not closed.

For the second assertion, let $\lambda \in \hat{I}$ and apply the first assertion to the singleton set $\{\lambda\}$.
□

Remark 5. A sequence in $X(\Gamma)$ could converge to more than one point. Indeed, let $\{(J, \gamma)\} \subset \hat{J}$, and $\gamma_n := \gamma \, \forall n \in \mathbb{N}$. Let $K \in \Sigma(\Gamma)$, such that $K \supset J$ and consider a character $(K, \chi) \in \hat{K}$ such that $\chi|_J = \gamma$. We show that $\gamma_n \to \chi$ in the upwards-looking topology for $X(\Gamma)$. Let $\mathcal{O}_{M,I}$ be any open neighbourhood of (K, χ) in $X(\Gamma)$, then $I \supset J$ and M is an open set in the point wise topology for $\hat{I}$ such that the set $M|_K := \{(K, \gamma|_K) : \gamma \in M\}$ contains (K, χ). Then $(J, \gamma) \in \mathcal{O}_{I,M}$, hence $\gamma_n \to \chi$ in the upwards-looking topology for $X(\Gamma)$. It is clear that we also have $\gamma_n \to \gamma$. Therefore there is more than one point of convergence of the sequence $\{\gamma_n\}$.

We found that a closed subset in the point wise topology on $\hat{\Gamma}$ is also closed in the upwards-looking topology on $X(\Gamma)$.

Proposition 6. *Suppose $F \subset \hat{\Gamma}$. If F is closed in the point wise topology on $\hat{\Gamma}$ then F is closed in the upwards-looking topology for $X(\Gamma)$.*

Proof. Let $\gamma \notin F$, we are going to show that $\gamma \notin \bar{F}^{ULT}$. We next consider two cases, i.e. when $\gamma \notin \hat{\Gamma}$ and when $\gamma \in \hat{\Gamma}$. If $\gamma \notin \hat{\Gamma}$, it is clear that $\gamma \notin \bar{F}^{ULT}$, because the restriction of every character in any open neighbourhood of γ will not give any character in $F \subset \hat{\Gamma}$. For the case $\gamma \in \hat{\Gamma}$, take $N = F^c$ as an open neighbourhood of γ in $\hat{\Gamma}$. Since $N \cap F = \emptyset$, then $\gamma \notin \bar{F}^{ULT}$. □

Corollary 7. *Let γ be any character in $\hat{\Gamma}$. Then the set $\{\gamma\}$ is closed in the upwards-looking topology for $X(\Gamma)$.*

For every order ideal $I \in \Sigma(\Gamma)$, we have $\hat{I} \subset X(\Gamma)$. Then it will be interesting to discuss the relative topology on every $\hat{I}$, and find out the relation with the point wise topology on it. We will write $\bar{F}^{RULT}$ for the closure of a subset $F \subset \hat{\Gamma}$ in the relative upwards-looking topology on $\hat{\Gamma}$, i.e

$$\bar{F}^{RULT} := \bar{F}^{ULT} \cap \hat{\Gamma}.$$

Theorem 8. *Suppose that Γ is a totally ordered abelian group. Then the point wise topology on $\hat{\Gamma}$ and the relative upward-looking topology on $\hat{\Gamma}$ are coincide.*

Proof. We firstly show that the point wise topology on $\hat{\Gamma}$ is stronger than the relative upwards-looking topology on $\hat{\Gamma}$. To show this, let $F \subset \hat{\Gamma}$ be closed under the relative upwards-looking topology on $\hat{\Gamma}$, i.e

$$\bar{F}^{RULT} = F.$$

We want to show that $\bar{F}^{PWT} = F$. It is clear that $F \subset \bar{F}^{PWT}$. Now let $\gamma \notin F$. An argumentation like in the proof of Proposition 6 implies that either $\gamma \notin \bar{F}^{ULT}$ or $\gamma \notin \hat{\Gamma}$. But $\gamma \in \hat{\Gamma}$, thus $\gamma \notin \bar{F}^{ULT}$. Therefore there is a neighbourhood N of γ such that for every $\chi \in N$, $(\Gamma, \chi) \notin F$. Hence $N \cap F = \emptyset$, which concludes that $\gamma \notin \bar{F}^{PWT}$.

Secondly we show that the relative upwards-looking topology in $\hat{\Gamma}$ is stronger than the point wise topology in $\hat{\Gamma}$. For this, let F be a closed subset in the point wise topology of $\hat{\Gamma}$, we want to show that $F = \bar{F}^{RULT}$. It is clear that $F \subset \bar{F}^{RULT}$. To complete the proof, let $\gamma \notin F$ and we show that $\gamma \notin \bar{F}^{ULT}$, which then implies that $\gamma \notin \bar{F}^{ULT} \cap \hat{\Gamma}$. If $\gamma \in \bar{F}^{ULT}$, then for every open neighbourhood $N(\gamma)$ in $\hat{\Gamma}$ of γ, there exists $\chi \in N(\gamma)$, an order ideal $J \subset \Gamma$ such that $(J, \chi|_J) \in F$. Since $F \subset \hat{\Gamma}$, it is clear that $J = \Gamma$. Thus $N(\gamma) \cap F \neq \emptyset$, which then implies that $\gamma \in \bar{F}^{PWT} = F$. This contradicts the assumption that $\gamma \notin F$. □

3. Characterization of Maximal Primitive Ideals

Let Γ be a totally ordered abelian group. The Toeplitz algebra $\mathcal{T}(\Gamma)$ of Γ is the C*-subalgebra of $B(\ell^2(\Gamma^+))$ generated by the isometries $\{T_x = T_x^\Gamma : x \in \Gamma^+\}$ which are defined in terms of the usual basis by $T_x(e_y) = e_{y+x}$. This algebra is universal for isometric representation of Γ^+ [5, Theorem 2.9].

Let I be an order ideal of Γ. Then the map $x \longmapsto T_{x+I}^{\Gamma/I}$ is an isometric representation of Γ^+ in $\mathcal{T}(\Gamma/I)$. Therefore by the universality of $\mathcal{T}(\Gamma)$, there is a homomorphism $Q_I : \mathcal{T}(\Gamma) \longrightarrow \mathcal{T}(\Gamma/I)$ such that $Q_I(T_x) = T_{x+I}^{\Gamma/I}$, and that Q_I is surjective. Suppose $\mathcal{C}(\Gamma, I)$ denotes the ideal in $\mathcal{T}(\Gamma)$ generated by $\{T_u T_u^* - T_v T_v^* : v - u \in I^+\}$ and $\mathrm{Ind}_{I^\perp}^{\hat{\Gamma}}(\mathcal{T}(\Gamma/I), \alpha^{\Gamma/I})$ is the closed subalgebra of $C(\hat{\Gamma}, \mathcal{T}(\Gamma/I))$ satisfying $f(xh) = \alpha_h^{\Gamma/I^{-1}}(f(x))$ for $x \in \hat{\Gamma}, h \in I^\perp$. It was proved in [1, Theorem 3.1] that there is a short exact sequence of C^*-algebras:

$$(3.1) \qquad 0 \to \mathcal{C}(\Gamma, I) \to \mathcal{T}(\Gamma) \xrightarrow{\phi_I} \mathrm{Ind}_{I^\perp}^{\hat{\Gamma}}(\mathcal{T}(\Gamma/I), \alpha^{\Gamma/I}) \to 0.$$

in which $\phi_I(a)(\gamma) = Q_I \circ (\alpha_\gamma^\Gamma)^{-1}(a)$ for $a \in \mathcal{T}(\Gamma)$, $\gamma \in \hat{\Gamma}$, and α_γ is dual action of $\hat{\Gamma}$ on $\mathcal{T}(\Gamma)$ characterized by $\alpha_\gamma^\Gamma(T_x) = \gamma(x)T_x$. The identity representation $T^{\Gamma/I}$ of $\mathcal{T}(\Gamma/I)$ is irreducible [5], it follows from [6, Proposition 6.16] that $\ker Q_I \circ (\alpha_\gamma^\Gamma)^{-1}$ is a primitive ideal of $\mathcal{T}(\Gamma)$. Moreover since

$$(3.2) \qquad Q_I \circ \alpha_\chi^\Gamma = \alpha_\chi^{\Gamma/I} \circ Q_I \text{ for } \chi \in I^\perp = \widehat{\Gamma/I},$$

the map $\gamma \longmapsto \ker Q_I \circ \alpha_\gamma^{-1}$ is constant on $I^\perp$ cosets in $\hat{\Gamma}$. Therefore it induces a well defined map L of $\hat{I} = \hat{\Gamma}/I^\perp$ into $\mathrm{Prim}\,\mathcal{T}(\Gamma)$. So that

$$(3.3) \qquad L(I, \gamma) := \ker Q_I \circ \alpha_\nu^{\Gamma^{-1}} \text{ where } \nu \in \hat{\Gamma} \text{ satisfies } \nu|_I = \gamma,$$

and then it was proved in Theorem 3.1 that L is a bijection of $X(\Gamma)$ onto $\mathrm{Prim}\,\mathcal{T}(\Gamma)$. Using the bijection $L : (I, \gamma) \in X(\Gamma) \longmapsto \ker Q_I \circ \alpha_\nu^{\Gamma^{-1}} \in \mathrm{Prim}\,\mathcal{T}(\Gamma)$, Adji and Raeburn describe a topology on $X(\Gamma)$ which corresponds to the hull-kernel topology on $\mathrm{Prim}\,\mathcal{T}(\Gamma)$.

Adji and Raeburn [2] proved that for any totally ordered abelian group Γ, every primitive ideal of $\mathcal{T}(\Gamma)$ has the form $\ker Q_I \circ (\alpha_\gamma^\Gamma)^{-1}$ for some $I \in \Sigma(\Gamma)$ and $\gamma \in \hat{\Gamma}$. In this section we are going to apply our findings and a theorem of Dixmier [4, 3.1.4], which states that a singleton subset $\{\mathcal{I}\}$ of the set of primitive ideals is closed if and only if $\mathcal{I}$ is a maximal ideal, to identify which ones are maximal ideals and which ones are not.

Theorem 4.7 of [2] says that for a totally ordered abelian group Γ such that $\Sigma(\Gamma)$ is isomorphic to a subset of $\mathbb{N} \cup \{\infty\}$, the topology of $\mathrm{Prim}\,\mathcal{T}(\Gamma)$ is described by the upwards-looking topology on the disjoint union $X(\Gamma) := \bigsqcup\{\hat{I} : I \in \Sigma(\Gamma)\}$ by specifying the closure operation. More recent result in [3, Theorem 3.1] shows that the topology on $\mathrm{Prim}\,\mathcal{T}(\Gamma)$ is given by the upwards-looking topology on $X(\Gamma)$, if and only if the set $\Sigma(\Gamma)$ of order ideals in Γ is well-ordered by inclusion.

Lemma 9. *Suppose that Γ is a totally ordered abelian group such that the chain $\Sigma(\Gamma)$ is well-ordered. If $I \in \Sigma(\Gamma)$ such that $I \neq \Gamma$, then for any $\gamma \in \hat{\Gamma}$, the primitive ideal*

$$\ker Q_I \circ (\alpha_\gamma^\Gamma)^{-1}$$

is not maximal.

Proof. The identification of primitive ideals in $\mathcal{T}(\Gamma)$ given in (3.3), implies that for any $I \in \Sigma(\Gamma)$ and $\gamma \in \hat{\Gamma}$, the singleton set $\{\ker Q_I \circ (\alpha_\gamma^\Gamma)^{-1}\}$ corresponds to the singleton set $\{\lambda\}$ for some $\lambda \in \hat{I}$ such that $\gamma|_I = \lambda$. From Proposition 4, the set $\{(I, \lambda)\}$ is not closed, hence the set $\{\ker Q_I \circ (\alpha_\gamma^\Gamma)^{-1}\}$ is not closed. Theorem of Dixmier hence implies that $\ker Q_I \circ (\alpha_\gamma^\Gamma)^{-1}$ is not a maximal ideal. $\qquad\qquad\square$

Lemma 10. *Suppose that Γ is a totally ordered abelian group such that the chain $\Sigma(\Gamma)$ is well-ordered. For any $\gamma \in \hat{\Gamma}$, the primitive ideal*

$$\ker Q_\Gamma \circ (\alpha_\gamma^\Gamma)^{-1}$$

is maximal.

Proof. Since $\{\ker Q_\Gamma \circ (\alpha_\gamma^\Gamma)^{-1}\}$ corresponds to $\{\gamma\}$ which is closed by Corollary 7, then $\ker Q_\Gamma \circ (\alpha_\gamma^\Gamma)^{-1}$ is maximal by the theorem of Dixmier. $\qquad\square$

Theorem 11. *Suppose that Γ is a totally ordered abelian group such that the chain $\Sigma(\Gamma)$ is well-ordered. Every maximal ideal in $\mathrm{Prim}\,\mathcal{T}(\Gamma)$ is of the form $\ker Q_\Gamma \circ (\alpha_\gamma^\Gamma)^{-1}$.*

Proof. By [2, Corollary 3.4], every primitive ideal in $\mathcal{T}(\Gamma)$ is of the form $\ker Q_I \circ (\alpha_\gamma^\Gamma)^{-1}$ for some $I \in \Sigma(\Gamma)$ and $\gamma \in \hat{\Gamma}$. The result then follows from Lemma 9 and Lemma 10. $\qquad\square$

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