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A NOTE ON COMMUTATOR IDEALS OF THE TWISTED TOEPLITZ ALGEBRAS OF ORDERED GROUPS

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Abstract. In this paper it is discussed that for the non-Archimedean ordered group $\Gamma = \mathbb{Z}\oplus_{\text{lex}}\mathbb{Z}$, the commutator ideal $\mathcal{C}(\Gamma,\sigma)$ is not simple. Further it is proved that the sufficient condition of the characterisation of commutator ideal of untwisted Toeplitz algebra apparently works for the twisted Toeplitz algebra $\mathcal{T}(\Gamma,\sigma)$. This result gives the sufficient condition of the characterisation of ideals of twisted Toeplitz algebras of totally ordered abelian groups.

1. Introduction

Generalising previous work of Coburn [3] on Toeplitz algebra of $\mathbb{Z}$, Douglas proved that the commutator ideal $\mathcal{C}(\Gamma)$ of Toeplitz algebra of subgroups Γ of $\mathbb{R}$, is simple. Further Ji and Xia in [7] proved that the ideal $\mathcal{C}(\Gamma)$ is the only nonzero simple ideal in the algebra $\mathcal{T}(\Gamma)$ of subgroups Γ of $\mathbb{R}$. In [9, 10] Murphy did another generalisation of this theory to a more general case: he considered Γ is a totally ordered abelian group.

A different way of generalisation was done by Ji [6], he employed a cocycle σ to the Toeplitz algebras of $\mathbb{R}$. This concept is later called the twisted Toeplitz algebra $\mathcal{T}(\Gamma,\sigma)$ of $\mathbb{R}$. Recently Rosjanuardi [12] has done a generalisation of the twisted Toeplitz algebras for totally ordered abelian groups Γ. His work was inspired by previous works of Ji, Murphy, Adji and her collaborators [6, 9, 2]. He proved that when the group Γ is Archimedean, then the commutator ideal $\mathcal{C}(\Gamma,\sigma)$ of $\mathcal{T}(\Gamma,\sigma)$ is simple. This result agrees with previous results given by Murphy [9] and Douglas [4]. In this paper is discussed that for the non-Archimedean ordered group $\Gamma = \mathbb{Z}\oplus_{\text{lex}}\mathbb{Z}$, the commutator ideal $\mathcal{C}(\Gamma,\sigma)$ is not simple. Further it is proved that the sufficient condition of the characterisation of commutator ideal given by Murphy [10] apparently works for the twisted Toeplitz algebra $\mathcal{T}(\Gamma,\sigma)$. This result gives the sufficient condition of the characterisation of ideals given by [12].

2. The Twisted Toeplitz Algebras of Totally Ordered Abelian Groups

Rosjanuardi [12] introduced the twisted Toeplitz algebras for totally ordered abelian groups that the basic idea is a generalisation of previous work of Ji [6]. Here we give a short overview of what he did.

2.1. **The twisted crossed product.** Suppose Γ is a totally ordered abelian group, and σ is a cocycle on Γ (i.e a mapping $\sigma : \Gamma\times\Gamma \longrightarrow \mathbb{T}$ satisfying $\sigma(x,y)\sigma(x+y,z) = \sigma(x,y+z)\sigma(y,z)$ for all x,y,z in Γ and $\sigma(x,0) = \sigma(0,x) = 1$ for all $x\in\Gamma$, where $\mathbb{T}$ is the unit circle $\{z\in\mathbb{C} : |z| = 1\}$). He defined a twisted dynamical system $(A,\Gamma^+,\alpha,\sigma)$ as a system consists of a C^*-algebra A, an action $\alpha : \Gamma^+ \to \text{Endo}(A)$ of Γ^+ on A by endomorphisms such that every α_t is extendible. A twisted covariant representation of $(A,\Gamma^+,\alpha,\sigma)$ is a pair (π,V) in which π is a nondegenerate representation of A, V is an isometric σ-representation of Γ^+, that is, $V_sV_t = \sigma(s,t)V_{s+t}$, and the covariant condition $\pi(\alpha_t(a)) = V_t\pi(a)V_t^*$ for $a\in A$ and $t\in\Gamma^+$ is satisfied.

He copy from [8] for the definition of a *twisted crossed product* of $(A,\Gamma^+,\alpha,\sigma)$. It is a triple (B,i_A,i_{Γ^+}) of a C^*-algebra B together with a nondegenerate homomorphism $i_A : A\to B$, a homomorphism $i_{\Gamma^+} : \Gamma^+ \to M(B)$ of Γ^+ into the multiplier algebra $M(B)$ of B such that $i_{\Gamma^+}(s)i_{\Gamma^+}(t) = \sigma(s,t)i_{\Gamma^+}(s+t)$ for all $s,t\in\Gamma^+$, and that satisfies the following conditions:

1) $i_A(\alpha_t(a)) = i_{\Gamma^+}(t) i_A(a) i_{\Gamma^+}(t)^*$ for $a \in A$ and $t \in \Gamma^+$;
2) for any covariant representation (π, V) of $(A, \Gamma^+, \alpha, \sigma)$, there is a nondegenerate representation $\pi \times V$ of B such that $(\pi \times V) \circ i_A = \pi$ and $\overline{(\pi \times V)} \circ i_{\Gamma^+} = V$;
3) B is generated by elements of the form $\{i_A(a) : a \in A\}$ and $\{i_{\Gamma^+}(x) : x \in \Gamma^+\}$.

In his remark he mentioned that the C^*-algebra B is the closure of

$$\mathrm{span}\{i_{\Gamma^+}(x) i_A(a) i_{\Gamma^+}(y)^* : x, y \in \Gamma^+, a \in A\}.$$

2.2. **The twisted Toeplitz algebra.** For each $y \in \Gamma^+$, he defined an operator $T_y : \ell^2(\Gamma^+) \to \ell^2(\Gamma^+)$ by

$$(T_y f)(x) = \begin{cases} \sigma(-x, y) f(x - y) & \text{if } x \geq y, \\ 0 & \text{else.} \end{cases} \tag{2.1}$$

This operator is an isometry which is non-unitary. He then defined that the twisted Toeplitz algebra $\mathcal{T}(\Gamma, \sigma)$ of totally ordered abelian groups Γ be the algebra generated by $\{T_y : y \in \Gamma^+\}$. This definition is a generalisation of previous one of Ji [6] for the case Γ is subgroups of real number $\mathbb{R}$, and also a generalisation to the twisted version of the untwisted version of Murphy [9] and Adji and her collaborators in [2] for totally ordered abelian groups.

Consider the C^*-subalgebra B_{Γ^+} of $\ell^\infty(\Gamma)$ generated by $\{1_x : x \in \Gamma^+\}$, where

$$1_x(y) = \begin{cases} 1 & \text{if } y \geq x, \\ 0 & \text{otherwise }. \end{cases}$$

The C^*-algebra B_{Γ^+} is unital with identity 1_0. For each x in Γ^+ the automorphism $\tau_x \in \mathrm{Aut}\, \ell^\infty(\Gamma)$ defined by $\tau_x(f)(y) = f(y - x)$ satisfies $\tau_x(1_y) = 1_{x+y}$. Therefore the restriction of τ on Γ^+ defines an action of Γ^+ by endomorphism of B_{Γ^+}. Suppose σ is a cocycle on Γ. In Lemma II. 10 he showed that the dynamical system $(B_{\Gamma^+}, \Gamma^+, \tau, \sigma)$ admits a nontrivial twisted covariant representation, and hence the crossed product $B_{\Gamma^+} \times_{\tau,\sigma} \Gamma^+$ exists. In Proposition II.11 he proved that this crossed product is universal for isometric σ-representation of Γ^+, and in Corollary II.15 he then showed that the twisted Toeplitz algebra is isomorphic to this crossed product. Hence this algebra is universal for isometric σ-representation of Γ^+, in the sense that if $\mathcal{V}$ is another C^*-algebra generated by an isometric σ-representation V of Γ^+, then there is a homomorphism $\phi : \mathcal{T}(\Gamma, \sigma) \longrightarrow \mathcal{V}$ such that $\phi(T_x) = V_x$. This ϕ is an isomorphism only if each V_x is non-unitary for every x in Γ^+.

Consider the subalgebra $B_{\Gamma^+,\infty}$ of B_{Γ^+} generated by $\{1_x - 1_y : y \geq x \in \Gamma^+\}$. In Corollary II.12 he showed that the crossed product $B_{\Gamma^+,\infty} \times_{\tau,\sigma} \Gamma^+$ is an ideal of $B_{\Gamma^+} \times_{\tau,\sigma} \Gamma^+$, and then he showed that $B_{\Gamma^+,\infty} \times_{\tau,\sigma} \Gamma^+$ is the commutator ideal $\mathcal{C}(\Gamma, \sigma)$ [12, Corollary II.18]. Recall that a totally ordered group Γ is called to be *Archimedean* if for every pair of elements x, y in Γ with $x, y > 0$ there exists a positive integer n such that $nx > y$. In Proposition II.19 he showed that if the group Γ is Archimedean, the commutator ideal $\mathcal{C}(\Gamma, \sigma) = \overline{\mathrm{span}}\{T_s(T_x T_x^* - T_y T_y^*) T_t : s, t \in \Gamma^+, x \leq y \in \Gamma^+\}$ is simple.

3. Ideals in the twisted Toeplitz algebra of $\mathbb{Z} \oplus_{\mathrm{lex}} \mathbb{Z}$

As we mentioned before that the simplicity of the commutator ideal $\mathcal{C}(\Gamma, \sigma)$ for Archimedean group Γ in [12] generalised previous work in the untwisted Toeplitz algebra of Douglas [4] on the Toeplitz algebra of subgroups of $\mathbb{R}$. Recall that groups with no elements of finite order are called torsion free groups, and thus an ordered group is torsion free. Murphy in [9, Theorem 4.3] proved the converse of Douglas's result, he proved that if Γ is a torsion free partially ordered abelian group for which the commutator ideal $\mathcal{C}(\Gamma)$ is simple, then Γ is Archimedean. These two results later then has a better perfection, Murphy in [10] proved that the commutator ideal of Toeplitz algebra of a totally ordered abelian group Γ is simple if and only if Γ is Archimedean.

We are currious whether this generality works for the twisted version. To find out the answer we need to investigate cases beyond the hypothesis of Murphy, for example we try to investigate Toeplitz algebras of a non-Archimedean group.

Suppose Γ is the group $\mathbb{Z} \oplus_{\mathrm{lex}} \mathbb{Z}$ with the **lexicographic** order:

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 < x_2, \text{ or } x_1 = x_2 \text{ and } y_1 \leq y_2.$$

Γ is a totally ordered group but not an Archimedean, because for $x = (1,0)$ and $y = (0,1)$ there is no $n \in \mathbb{N}$ such that $x \leq ny$. The set $I := \{(0,y) : y \in \mathbb{Z}\}$ is an order ideal, because $0 \leq (x,z) \leq (0,y)$ implies that $x = 0$. Moreover it is the only proper ideal of Γ.

Now let $C_{I^+} = \overline{\operatorname{span}}\{1_x - 1_y : x, y \in \Gamma^+, y - x \in I^+\}$. Proposition 8.1.4 of [1] implies that C_{I^+} s an extendibly τ-invariant ideal of $B_{\Gamma^+,\infty}$. This lemma will be very important in sequel.

Lemma 1. *Suppose* $\Gamma = \mathbb{Z} \oplus_{\text{lex}} \mathbb{Z}$ *and* $\omega_\theta((m,n),(p,q)) = e^{2\pi i\theta np}$ *for* $\theta \in [0,1)$ *is a cocycle on* Γ*. The ideal* $\mathcal{C}(\Gamma, I, \omega_\theta)$ *generated by* $\{1 - T_x T_x^* : x \in I^+\}$ *is isomorphic to the crossed product* $C_{I^+} \times_{\tau,\omega_\theta} \Gamma^+$*.*

Proof. Applying Theorem II.20 of [12] to the ideal C_{I^+} shows that $C_{I^+} \times_{\tau,\omega_\theta} \Gamma^+$ is an ideal of the crossed product $(B_{\Gamma^+} \times_{\tau,\omega_\theta} \Gamma^+, i_{B_{\Gamma^+}}, i_{\Gamma^+})$, and is isomorphic to

$$\overline{\operatorname{span}}\{i_{\Gamma^+}(s)^* i_{B_{\Gamma^+}}(1_x - 1_y) i_{\Gamma^+}(t) : s,t,x,y \in \Gamma^+, y - x \in I^+\}. \tag{3.1}$$

Applying Proposition II.11 of [12] to $T : \Gamma^+ \longrightarrow B(l^2(\Gamma^+))$ gives a homomorphism

$$\pi_T \times T : B_{\Gamma^+} \times_{\tau,\omega_\theta} \Gamma^+ \longrightarrow \mathcal{T}(\Gamma, \omega_\theta) \tag{3.2}$$

such that $\pi_T \times T(i_{\Gamma^+}(x)) = T_x$. Theorem II.11 of [12] implies that $\pi_T \times T$ is injective. Since $\{i_{\Gamma^+}(x) : x \in \Gamma^+\}$ generates $B_{\Gamma^+} \times_{\tau,\omega_\theta} \Gamma^+$ (by [12, Proposition II.11]), $\pi_T \times T$ is an isomorphism of $B_{\Gamma^+} \times_{\tau,\omega_\theta} \Gamma^+$ onto $\mathcal{T}(\Gamma, \omega_\theta) := C^*(\{T_x : x \in \Gamma^+\})$. The image of (3.1) is

$$\overline{\operatorname{span}}\{T_s^*(T_x T_x^* - T_y T_y^*)T_t : s,t,x,y \in \Gamma^+, y - x \in I^+\}, \tag{3.3}$$

which is therefore the ideal $\mathcal{C}(\Gamma, I, \omega_\theta)$ by [12, Lemma III.6] . □

Lemma 2. *Suppose* $\Gamma = \mathbb{Z} \oplus_{\text{lex}} \mathbb{Z}$ *and* $\omega_\theta((m,n),(p,q)) = e^{2\pi i\theta np}$ *for* $\theta \in [0,1)$ *is a cocycle on* Γ*. There is a short exact sequence*

$$0 \longrightarrow C_{I^+} \times_{\tau,\omega_\theta} \Gamma^+ \longrightarrow B_{\Gamma^+,\infty} \times_{\tau,\omega_\theta} \Gamma^+ \longrightarrow B_{(\Gamma/I)^+} \times_{\tilde{\tau},\omega_\theta} \Gamma^+ \longrightarrow 0$$

Proof. Theorem II.20 of [12] implies that there is a short exact sequence

$$0 \longrightarrow C_{I^+} \times_{\tau,\omega_\theta} \Gamma^+ \longrightarrow B_{\Gamma^+,\infty} \times_{\tau,\omega_\theta} \Gamma^+ \longrightarrow (B_{\Gamma^+,\infty}/C_{I^+}) \times_{\tilde{\tau},\omega_\theta} \Gamma^+ \longrightarrow 0.$$

Since Lemma 8.1.5 of [1] implies that

$$B_{\Gamma^+,\infty}/C_{I^+} \cong B_{(\Gamma/I)^+},$$

the lemma just has been proved.

□

Proposition 3. *Suppose* $\Gamma = \mathbb{Z} \oplus_{\text{lex}} \mathbb{Z}$ *and* $\omega_\theta((m,n),(p,q)) = e^{2\pi i\theta np}$ *for* $\theta \in [0,1)$ *is a cocycle on* Γ*, then the commutator ideal* $\mathcal{C}(\Gamma, \omega_\theta)$ *is not simple.*

Proof. Let I be the ideal $\{(0,y) : y \in \mathbb{Z}\}$ of Γ, and Lemma III. 6 of [12] implies that $\mathcal{C}(\Gamma, I, \omega_\theta)$ is an ideal in $\mathcal{C}(\Gamma, \omega_\theta)$. Since $T_x T_x^* = 0$ only for $x = 0$, we have $1 - T_x T_x^* \neq 0$ for x in I^+, and hence $\mathcal{C}(\Gamma, I, \omega_\theta)$ is nonzero.

Next we claim that $\mathcal{C}(\Gamma, I, \omega_\theta) \neq \mathcal{C}(\Gamma, \omega_\theta)$. Suppose the contrary. Corollary II.18 of [12] says that $\mathcal{C}(\Gamma, \omega_\theta) \cong B_{\Gamma^+,\infty} \times_{\tau,\omega_\theta} \Gamma^+$, hence Lemma 1 and Lemma 2 implies that

$$\mathcal{C}(\Gamma, \omega_\theta)/\mathcal{C}(\Gamma, I, \omega_\theta) \cong B_{(\Gamma/I)^+} \times_{\tilde{\tau},\omega_\theta} \Gamma^+.$$

Hence $B_{(\Gamma/I)^+} \times_{\tilde{\tau},\omega_\theta} \Gamma^+ = 0$. On the other sides, this contradicts the main result of [13] which says that

$$B_{(\Gamma/I)^+} \times_{\tilde{\tau},\omega_\theta} \Gamma^+ \cong \mathcal{K}(\ell^2(\mathbb{N})) \otimes C^*(I),$$

which is nonzero. Therefore $\mathcal{C}(\Gamma, I, \omega_\theta) \neq \mathcal{C}(\Gamma, \omega_\theta)$, and thus the ideal $\mathcal{C}(\Gamma, \omega_\theta)$ is not simple. □

Proposition 4. *Suppose* $\Gamma = \mathbb{Z} \oplus_{\text{lex}} \mathbb{Z}$ *and* $\omega_\theta((m,n),(p,q)) = e^{2\pi i\theta np}$ *for* $\theta \in [0,1)$ *is a cocycle on* Γ*. The ideal* $\mathcal{C}(\Gamma, I, \omega_\theta)$ *is simple.*

Proof. The restriction $\operatorname{res}\omega_\theta$ of the cocycle ω_θ on $I \times I$ is trivial, hence the Toeplitz algebra $\mathcal{T}(I, \operatorname{res}\omega_\theta) = \mathcal{T}(I)$. Since I is Archimedean, [12, Proposition II.19] implies that the commutator ideal $\mathcal{C}(I, \operatorname{res}\omega_\theta) = \mathcal{C}(I)$ in $\mathcal{T}(I)$ is simple. Since $\mathcal{C}(I)$ is Morita equivalent to $\mathcal{C}(\Gamma, I, \omega_\theta)$ [12, Theorem III.9], the Rieffel correspondence [11, Theorem 3.22] implies the result. □

4. The Sufficient Condition for Simple Commutator Ideals

Suppose I is a subgroup of an abelian group Γ. One way to construct a cocycle on Γ is to inflate one of quotient group Γ/I to a cocycle on Γ. The procedure is as follow. Let $q : \Gamma \longrightarrow \Gamma/I$ canonical quotient homomorphism, and

$$\begin{array}{rcl} q \times q : \Gamma \times \Gamma & \longrightarrow & \Gamma/I \times \Gamma/I \\ (x, y) & \longmapsto & (q(x), q(y)) \end{array}$$

If σ is a cocycle on Γ/I, $\sigma \circ (q \times q)$ defines a cocycle Γ, and we write $\sigma \circ (q \times q) = \inf \sigma$. The cocycle $\inf \sigma$ is called an *inflated cocycle* from σ. The cocycle $\inf \sigma$ is trivial on I, because for every $x, y \in I$ we have

$$\inf \sigma(x, y) = \sigma \circ (q \times q)(x, y) = \sigma(q(x), q(y)) = \sigma(0, 0) = 1. \tag{4.1}$$

The following proposition gives the sufficient condition for simplicity of the twisted commutator ideal of the twisted Toeplitz algebra. The spirit of proof is basically that of Murphy [9].

Proposition 5. *Suppose I is an order ideal of a totally ordered abelian group Γ, and σ is a cocycle on Γ/I. If the commutator ideal $\mathcal{C}(\Gamma, \inf \sigma)$ is simple then Γ is Archimedean.*

Proof. Suppose I is an ideal of Γ, since a group is Archimedean if and only if it is simple [5, p. 47], we claim that $I = 0$ or either $I = \Gamma$. Lemma III.2 of [12] implies that there is a surjection $\phi : \mathcal{T}(\Gamma, \inf \sigma) \longrightarrow \mathcal{T}(\Gamma/I, \sigma)$ such that $\phi(T_x) = T^{\Gamma/I}_{q(x)}$ where $T^{\Gamma/I}$ are generators of $\mathcal{T}(\Gamma/I, \sigma)$, and $q : \Gamma \longrightarrow \Gamma/I$ is the canonical quotient. Hence the restriction $\phi|_{\mathcal{C}(\Gamma, \inf \sigma)}$ of ϕ on $\mathcal{C}(\Gamma, \inf \sigma)$ is also a surjection. Since $\mathcal{C}(\Gamma, \inf \sigma)$ is simple, $\ker \phi|_{\mathcal{C}(\Gamma, \inf \sigma)}$ is zero or either the whole $\mathcal{C}(\Gamma, \inf \sigma)$, on the other word $\phi|_{\mathcal{C}(\Gamma, \inf \sigma)}$ is injective or either a zero homomorphism. In the first case, let x in I^+, and we compute

$$\phi|_{\mathcal{C}(\Gamma, \inf \sigma)}(1 - T_x T_x^*) = 1^{\Gamma/I} - T^{\Gamma/I}_{q(x)} T^{\Gamma/I^*}_{q(x)} = 0.$$

Then $1 - T_x T_x^* = 0$ by injectivity, hence $x = 0$ because $T_x T_x^* = 1$ only if $x = 0$. Therefore $I = 0$. In the second case, surjectivity of $\phi|_{\mathcal{C}(\Gamma, \inf \sigma)}$ implies that $\mathcal{C}(\Gamma/I, \sigma) = 0$. Hence $\Gamma/I = 0$, and therefore $I = \Gamma$. □

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