TWISTED SEMIGROUP CROSSED PRODUCTS AND THE TWISTED TOEPLITZ ALGEBRAS OF ORDERED GROUPS

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Abstract. Let Γ^+ be the positive cone of a totally ordered abelian group Γ, and σ a cocycle in Γ. We study the twisted crossed products by actions of Γ^+ as endomorphisms of C^*-algebras, and use this to generalized the theorem of Ji.

1. Introduction

Theory of crossed products by semigroups of endomorphisms has been successfully used to study the Toeplitz algebras [3, 2, 9]. Having the success, it is natural to extend the ideas into more general contexts. One direction is to the twisted version, in which the semigroup is implemented as a projective isometric representation by a given scalar valued cocycle. This has been done by Laca in [7] for a class of semigroups in real numbers and for unital C^*-algebras. He introduced briefly the definition of twisted semigroup crossed products similar to the way in which the untwisted version is defined, by way of a universal property with respect to twisted covariant representations, and then he stated a theorem about characterization of faithful representations of this crossed products as in [2]. A wide range of twisted semigroups crossed products have also been studied by Fowler and Raeburn in [5], but (again) they work with unital algebras.

Here we set up a theory of twisted semigroup crossed product for a class of semigroups, which are the positive cones Γ^+ in an arbitrary totally ordered abelian groups Γ, and for nonunital C^*-algebras. This setting is nicely fit to study the Toeplitz algebras $\mathcal{T}_\Gamma^\sigma$ of noncommutative tori which was used by Ji in [6]. He worked with a dense subgroup Γ of real numbers. Our goal is to generalize Ji's theorem. Phillips and Raeburn have showed a way to do this [11], they gave an alternative proof of Ji's theorem by employing a dilation technique. We choose to deal directly with the twisted semigroup crossed product.

In the first section we begin with the definition of twisted semigroup crossed products where the semigroups act by endomorphisms on nonunital C^*-algebras, it is required that all these endomorphisms have to be extendible. The construction and the uniqueness of such crossed products can be done by the same method as in the untwisted version [2, 10]. We have a theorem about the short exact sequences of twisted semigroups crossed products, which is similar to those in [2, 10]. In Section 2 we import results from [5] to have a twisted version of [3]: the universal C^*-algebra

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$C^*(\Gamma^+, \sigma)$ for isometric σ-representation is a twisted semigroup crossed product, and that the C^*-algebras generated by two nonunitary isometric σ-representations are canonically isomorphic. In the final section we give an application to the Toeplitz algebra $\mathcal{T}_\Gamma^\sigma$. Our results extend Ji's theorem to abelian totally ordered groups.

We can see from this work that many of the results about semigroups crossed products and the untwisted Toeplitz algebras extend to the twisted version in very natural way. This rises a question wether $\mathcal{T}_\Gamma^\sigma$ has a similar structure to the untwisted Toeplitz algebra.

2. Twisted semigroup crossed products

Let Γ be a totally ordered discrete abelian group with positive cone Γ^+. A *cocycle* σ on Γ is a function $\sigma : \Gamma \times \Gamma \to \mathbb{T}$ satisfies: $\sigma(x, 0) = 1 = \sigma(0, x)$ for $x \in \Gamma$, and $\sigma(x, y)\sigma(x + y, z) = \sigma(x, z + y)\sigma(y, z)$ for $x, y, z \in \Gamma$. We recall from [1] the notion about extendible endomorphism. An endomorphism ϕ on a nonunital C^*-algebra A is *extendible* if it extends uniquely to a strictly continuous endomorphism $\overline{\phi}$ of the multiplier algebra $M(A)$, this happens precisely when there is an approximate identity (e_λ) for A and a projection p_ϕ in $M(A)$ such that $\phi(e_\lambda)$ converges strictly to p_ϕ in $M(A)$. An extendible endomorphism satisfies that $\phi(e_\lambda)$ converges to $\overline{\phi}(1_{M(A)})$ strictly in $M(A)$ for any approximate identity $(e_\lambda) \subset A$. Every endomorphism on a unital C^*-algebra is trivially extendible.

A *semigroup dynamical system* is a triple $(A, \Gamma^+, \alpha, \sigma)$ consisting of a C^*-algebra A which may not be unital, a cocycle σ, and an action α of Γ^+ on A by extendible endomorphisms in the sense that each α_x is extendible. An *isometric σ-representation* of Γ^+ is a map V of Γ^+ into the set of isometries $\mathrm{Isom}(H)$ on a Hilbert space H such that $V_xV_y = \sigma(x, y)V_{x+y}$. A *covariant representation* of $(A, \Gamma^+, \alpha, \sigma)$ is a pair (π, V) in which π is a nondegenerate representation of A on a Hilbert space H, and V is an isometric σ-representation of Γ^+ such that $\pi(\alpha_x(a)) = V_x\pi(a)V_x^*$ for $a \in A$ and $x \in \Gamma^+$.

A *twisted crossed product* for $(A, \Gamma^+, \alpha, \sigma)$ is a C^*-algebra B together with a nondegenerate homomorphism $i_A : A \to B$ and a twisted embedding of Γ^+ as isometries $i_{\Gamma^+} : \Gamma^+ \to M(B)$ which satisfies

1. covariance relation: $i_A(\alpha_x(a)) = i_{\Gamma^+}(x)i_A(a)i_{\Gamma^+}(x)^*$ for $a \in A$ and $x \in \Gamma^+$.
2. for any other covariant representation (π, V) of $(A, \Gamma^+, \alpha, \sigma)$, there is a nondegenerate representation $\pi \times V$ of B such that $(\pi \times V) \circ i_A = \pi$ and $\overline{(\pi \times V)} \circ i_{\Gamma^+} = V$
3. B is generated by $\{i_A(a)i_{\Gamma^+}(x) : a \in A, x \in \Gamma^+\}$.

Lemma 2.1. *The C^*-algebra B is* $\overline{\mathrm{span}}\{i_{\Gamma^+}(x)^*i_A(a)i_{\Gamma^+}(y) : a \in A, x, y \in \Gamma^+\}$.

Proof. For short, we write $i_{\Gamma^+}(x)$ as $i(x)$. We only have to show that the set $\{i(x)^*i_A(a)i(y) : a \in A, x, y \in \Gamma^+\}$ is closed under multiplication. Fix $i(x)^*i_A(a)i(y)$ and $i(s)^*i_A(b)i(t)$. Assume that $y \leq s$, and do similar computations for $y > s$, as follow:

$$\begin{aligned}
& i(x)^* i_A(a) i(y) i(s-y+y)^* i_A(b) i(t) \\
&= i(x)^* i_A(a) i(y) (\overline{\sigma(s-y,y)} i(s-y) i(y))^* i_A(b) i(t) \\
&= \sigma(s-y,y) i(x)^* i_A(a) i(y) i(y)^* i(s-y)^* i_A(b) i(t) \\
&= \sigma(s-y,y) i(x)^* i_A(a) \overline{i_A}(\overline{\alpha}_y(1)) i(s-y)^* i_A(b) i(t) \\
&= \sigma(s-y,y) i(x)^* i_A(a\overline{\alpha}_y(1)) i(s-y)^* i_A(b) i(t) \\
&= \sigma(s-y,y) i(x)^* i(s-y)^* i_A(\alpha_{s-y}(a\overline{\alpha}_y(1))) i_A(b) i(t) \\
&= \sigma(s-y,y) \overline{\sigma(s-y,x)} i(s-y+x)^* i_A(\alpha_{s-y}(a\overline{\alpha}_y(1)) b) i(t). \qquad (2.1)
\end{aligned}$$

□

Proposition 2.2. *If a dynamical system $(A, \Gamma^+, \alpha, \sigma)$ has a nonzero covariant representation, then there exists a crossed product for the system, and it is unique up to isomorphism.*

Proof. The proof is exactly the same method as in the untwisted version [1, 10]. A covariant representation (π, V) on H is *cyclic* if the C^*-algebra generated by $\pi(A) \cup V(\Gamma^+)$ has a cyclic vector. Two cyclic representations are equivalent if there is a unitary intertwining their images.

Every covariant representation (π, V) of $(A, \Gamma^+, \alpha, \sigma)$ is equivalent to a direct sum of cyclic covariant representations. Take a set $\mathcal{S}$ of cyclic covariant representation with the property that every cyclic covariant representation of the system is equivalent to an element of $\mathcal{S}$. Define $i_A(a) = \oplus_{(\pi,V)\in\mathcal{S}} \pi(a)$ for $a \in A$, and $i_{\Gamma^+}(x) = \oplus_{(\pi,V)\in\mathcal{S}} V_x$ for $x \in \Gamma^+$. Then the C^*-algebra B generated by $i_A(A) \cup i_{\Gamma^+}(\Gamma^+)$, together with i_A and i_{Γ^+} is a crossed product for the system.

It is unique because if (C, j_A, j_{Γ^+}) is another crossed product for the system, then the homomorphism $j_A \times j_{\Gamma^+} : B \to C$ from part (2) of the definition is an isomorphism such that $(j_A \times j_{\Gamma^+}) \circ i_A = j_A$ and $(\overline{j_A \times j_{\Gamma^+}}) \circ i_{\Gamma^+} = j_{\Gamma^+}$. □

Remark 2.3. Denote *the* crossed product for $(A, \Gamma^+, \alpha, \sigma)$ by $A \times_{\alpha,\sigma} \Gamma^+$. If $(A, \Gamma^+, \alpha, \sigma)$ and $(B, \Gamma^+, \beta, \sigma)$ are two dynamical systems, and $\psi : A \to B$ is an isomorphism such that $\psi(\alpha_z(a)) = \beta_z(\psi(a))$ for all $a \in A$ and $z \in \Gamma^+$, then $A \times_{\alpha,\sigma} \Gamma^+$ is isomorphic to $B \times_{\beta,\sigma} \Gamma^+$.

Lemma 2.4. *Suppose $(A, \Gamma^+, \alpha, \sigma)$ and $(A, \Gamma^+, \alpha, \omega)$ are two dynamical system, which have nontrivial covariant representations, and that $[\sigma] = [\omega]$ in $H^2(\Gamma, \mathbb{T})$. Then the crossed product $A \times_{\alpha,\sigma} \Gamma^+$ is isomorphic to $A \times_{\alpha,\omega} \Gamma^+$.*

Proof. Since $[\sigma] = [\omega]$, there is a function ν of Γ to $\mathbb{T}$ such that $\nu(0) = 1$, and that $\sigma(x, y) = \frac{\nu(x+y)}{\nu(x)\nu(y)} \omega(x, y)$. Let i_A and i_{Γ^+} be the canonical embedding of A and Γ^+ into $M(A \times_{\alpha,\sigma} \Gamma^+)$. If $j_{\Gamma^+} : \Gamma^+ \to M(A \times_{\alpha,\sigma} \Gamma^+)$, which is given by $j_{\Gamma^+}(x) = \nu(x) i_{\Gamma^+}(x)$. Then $j_{\Gamma^+}(x)^* j_{\Gamma^+}(x) = 1$ for all $x \in \Gamma^+$, and

$$\begin{aligned}
j_{\Gamma^+}(x) j_{\Gamma^+}(y) &= \nu(x)\nu(y)\sigma(x, y) j_{\Gamma^+}(x + y) \\
&= \frac{\nu(x)\nu(y)}{\nu(x + y)} \sigma(x, y) i_{\Gamma^+}(x + y) \\
&= \omega(x, y) i_{\Gamma^+}(x + y) \quad \text{for } x, y \in \Gamma^+.
\end{aligned}$$

So $(A \times_{\alpha,\sigma} \Gamma^+, i_A, j_{\Gamma^+})$ is a crossed product for $(A, \Gamma^+, \alpha, \omega)$. Therefore $A \times_{\alpha,\sigma} \Gamma^+$ is isomorphic to $A \times_{\alpha,\omega} \Gamma^+$. $\square$

We recall again from [1] the definition of extendible invariant ideals in C^*-algebras. Suppose α is an extendible endomorphism of a C^*-algebra A. Let I be an ideal of A, and $\psi : A \to M(I)$ the canonical nondegenerate homomorphism defined by $\psi(a)i = a.i$ for $a \in A$ and $i \in I$. It is α-invariant if $\alpha(I) \subset I$. An α-invariant ideal I is called *extendibly α-invariant ideal* if for an approximate identity $(u_\lambda) \subset I$ we have $\alpha(u_\lambda)$ converges strictly to $\overline{\psi}(\overline{\alpha}(1_{M(A)}))$ in $M(I)$.

Now given a dynamical system $(A, \Gamma^+, \alpha, \sigma)$ by extendible endomorphisms, and an ideal I of A, it was proved in [1] that there is a system $(A/I, \Gamma^+, \tilde{\alpha}, \sigma)$ with extendible endomorphisms $\tilde{\alpha}_x(aI) = \alpha_x(a)I$ for all $a \in A$ and $x \in \Gamma^+$.

Theorem 2.5.
Suppose $(A, \Gamma^+, \alpha, \sigma)$ is a system with extendible endomorphism, and I is an extendible invariant ideal of A. Let $(A \times_{\alpha,\sigma} \Gamma^+, \iota_A, j_{\Gamma^+})$ be the crossed product for $(A, \Gamma^+, \alpha, \sigma)$. Then there exists a short exact sequence

$$0 \longrightarrow I \times_{\alpha,\sigma} \Gamma^+ \xrightarrow{\Phi} A \times_{\alpha,\sigma} \Gamma^+ \xrightarrow{\Psi} A/I \times_{\tilde{\alpha},\sigma} \Gamma^+ \longrightarrow 0, \tag{2.2}$$

where Φ is an isomorphism of $I \times_{\alpha,\sigma} \Gamma^+$ onto the ideal

$$D := \overline{\operatorname{span}}\{j_{\Gamma^+}(x)^* \iota_A(a) j_{\Gamma^+}(y) : x, y \in \Gamma^+, i \in I\} \text{ of } A \times_{\alpha,\sigma} \Gamma^+.$$

Proof. The same proof of untwisted version in [1, Theorem 3.1] and [10, Theorem1.7] works. By doing the similar computations to 2.1 on the generators, we see that D is an ideal of $A \times_{\alpha,\sigma} \Gamma^+$. Let $\beta : A \times_{\alpha,\sigma} \Gamma^+ \to M(D)$ be the nondegenerate canonical homomorphism. Then $(D, \beta \circ \iota_A|_I, \overline{\beta} \circ j_{\Gamma^+})$ is a crossed product for $(I, \Gamma^+, \alpha, \sigma)$: $\beta \circ \iota_A|_I$ is nondegenerate because I is an extendible ideal. So there is an isomorphism $\Phi := \beta \circ \iota_A|_I \times \overline{\beta} \circ j_{\Gamma^+}$ which gives the first half of (2.2).

Now the homomorphism $\Psi : A \times_{\alpha,\sigma} \Gamma^+ \to A/I \times_{\tilde{\alpha},\sigma} \Gamma^+$ that satisfies the second half would be $\iota_{A/I} \circ q \times k_{\Gamma^+}$ where q is the quotient map and $(\iota_{A/I},\ k_{\Gamma^+})$ are the canonical homomorphisms of A/I and Γ^+ respectively into the crossed product $A/I \times_{\tilde{\alpha},\sigma} \Gamma^+$. The kernel of $\iota_{A/I} \circ q \times k_{\Gamma^+}$ certainly contains D. To see that it is D, take a representation ρ of $A \times_{\alpha,\sigma} \Gamma^+$ with kernel D. Since $\ker(\rho \circ \iota_A)$ is I, there is a nondegenerate representation η of A/I such that the pair $(\eta, \overline{\rho} \circ j_{\Gamma^+})$ is a nontrivial covariant representation of $(A/I, \Gamma^+, \tilde{\alpha}, \sigma,)$. Therefore $A/I \times_{\tilde{\alpha},\sigma} \Gamma^+$ exists, and there is a nondegenerate representation $\Theta := \eta \times \overline{\rho} \circ j_{\Gamma^+}$ of $A/I \times_{\tilde{\alpha},\sigma} \Gamma^+$ such that $\Theta \circ \Psi = \rho$. Thus $\ker \Psi \subset D$. $\square$

3. The system $(B_{\Gamma^+}, \Gamma^+, \tau, \sigma)$

Suppose Γ is a totally ordered abelian group with positive cone Γ^+, and let σ be a cocycle on Γ. Consider the C^*-subalgebra B_{Γ^+} of $\ell^\infty(\Gamma)$ spanned by the functions $\{1_x \in \ell^\infty(\Gamma) : x \in \Gamma^+\}$, where

$$1_x(y) = \begin{cases} 1 & \text{if } y \geq x, \\ 0 & \text{otherwise .} \end{cases}$$

For each $x \in \Gamma^+$, the left translation on $\ell^\infty(\Gamma)$ restricts to an action τ of Γ^+ by endomorphisms on B_{Γ^+} such that $\tau_x(1_y) = 1_{y+x}$. Since B_{Γ^+} has a unit 1_0, hence every endomorphism τ_x is extendible. The system $(B_{\Gamma^+}, \Gamma^+, \tau, \sigma)$ has a nontrivial covariant representation, which can be constructed directly or use [8, Remark 2.5] alternatively.

Fowler and Raeburn showed in [5] that the C^*-algebra $B_{\Gamma^+} \times_{\tau,\sigma} \Gamma^+$ is one of the crossed product by a product system. Their results in §4 implies that $B_{\Gamma^+} \times_{\tau,\sigma} \Gamma^+$ has a property which is an analogue to the untwisted version in [3]. It is universal which characterizes the C^*-algebra $C^*(\Gamma^+, \sigma)$: every isometric σ-representation V of Γ^+ gives a faithful representation of $B_{\Gamma^+} \times_{\tau,\sigma} \Gamma^+$ precisely when all of V_x are nonunitary. We want to discuss and translate the theorems into our setting.

3.1. **Discrete product system over Γ^+.** A *discrete product system E over Γ^+* is a family of complex finite dimension Hilbert spaces $\{E_x : x \in \Gamma^+\}$ on which there is an operation satisfying:

(i) for each $x, y \in \Gamma^+$, there is a bilinear map $(u, v) \in E_x \times E_y \mapsto uv \in E_{x+y}$ such that $E_x E_y$ spans a dense subset of E_{x+y};
(ii) associativity: $(uv)w = u(vw)$ for every $u, v, w \in E$; and
(iii) $(uu'|vv') = (u|v)(u'|v')$ whenever $u, v \in E_x$ and $u', v' \in E_y$.

Note that (i) and (iii) imply that the map $u \otimes v \in E_x \otimes E_y \mapsto uv \in E_{x+y}$ extends to a unitary operator from $E_x \otimes E_y$ onto E_{x+y}.

Example 3.1. Let E_x be the Hilbert space of complex numbers $\mathbb{C}$ for all $x \in \Gamma^+$, then $\{E_x :\in \Gamma^+\}$ is a product system with the operation defined by $(u, v) \in E_x \times E_y$ maps to the usual multiplication of complex numbers $uv \in E_{x+y}$. It is denoted by $\Gamma^+ \times \mathbb{C}$. Now given a cocycle σ on Γ. Then a family of Hilbert spaces of complex numbers $\{E_x :\in \Gamma^+\}$ becomes another product system, with the twisted multiplication: $(u, v) \in E_x \times E_y \mapsto \sigma(x, y)uv \in E_{x+y}$. This system is called the product system $\Gamma^+ \times \mathbb{C}$ twisted by σ, and is denoted by $(\Gamma^+ \times \mathbb{C})^\sigma$.

A *representation* of a discrete product system E on a Hilbert space H is an operator valued map $\phi : E \to B(H)$ such that

(i) $\phi(uv) = \phi(u)\phi(v)$ for $u \in E_x$ and $v \in E_y$
(ii) $\phi(v)^*\phi(u) = (u|v)I$ whenever u and v contained in the same E_x.

We use notation ϕ_x for the restriction of ϕ to the fiber E_x. Thus the condition (i) means that $\phi_x(u)\phi_y(v) = \phi_{x+y}(uv)$ for $u \in E_x$ and $v \in E_y$. It was shown in [4, p.8] that each of ϕ_x is a linear map.

Lemma 3.2. *A representation ϕ of the product system $(\Gamma^+ \times \mathbb{C})^\sigma$ corresponds to an isometric σ-representation $x \in \Gamma^+ \mapsto \phi_x(1) \in B(H)$ of Γ^+.*

Proof. Suppose ϕ is a representation of $E = (\Gamma^+ \times \mathbb{C})^\sigma$. Then for every $x \in \Gamma^+$, the linear map $\phi_x(1)$ is an isometry because $(\phi_x(1))^*\phi_x(1) = (1|1)I = I$, and $I = \phi_0(1)^*\phi_0(1) = \phi_0(1)^*\phi_0(1)\phi_0(1) = \phi_0(1)$. Moreover $\phi_x(1)\phi_y(1) = \phi_{x+y}(\sigma(x, y)1) = \sigma(x, y)\phi_{x+y}(1)$ for $x, y \in \Gamma^+$. Therefore the map $x \in \Gamma^+ \mapsto \phi_x(1) \in B(H)$ is an isometric σ-representation of Γ^+.

Conversely, suppose $V : \Gamma^+ \to B(H)$ is an isometric σ-representation of Γ^+. Then let $\phi : E \to B(H)$ be defined by $\phi(\lambda) := \lambda V_x$ for $\lambda \in E_x$. It is a representation of E:

$\phi_x(\lambda)\phi_y(\eta) = \lambda\eta V_x V_y = \sigma(x,y)\lambda\eta V_{x+y} = \phi_{x+y}(\sigma(x,y)\lambda\eta)$ for $\lambda \in E_x$ and $\eta \in E_y$, and $\phi_x(\lambda)^*\phi_x(\eta) = \overline{\lambda}\eta V_x^* V_x = (\eta|\lambda)I$. □

Remark 3.3. It was proved in [4, Proposition 2.7] that a representation $\phi : E \to B(H)$ of a product system E induces an action $\gamma^\phi : \Gamma^+ \to \mathrm{end}(B(H))$ of Γ^+ on $B(H)$ by endomorphisms such that

$$\phi(E_x) = \{T \in B(H) : \gamma_x^\phi(S)T = TS \text{ for each } S \in B(H)\},$$

and if $\{u_i : i = 1, 2, \cdots, n\}$ is an orthonormal basis for E_x then the endomorphism γ_x^ϕ is given by the sum

$$\gamma_x^\phi(S) = \sum_{i=1}^{n} \phi_x(u_i) S \phi(u_i)_x^* \text{ for } S \in B(H). \tag{3.1}$$

3.2. **Semigroup crossed products by product systems.** Consider a system (A, Γ^+, α, E) consists of a unital C^*-algebra A, an action α of Γ^+ on A by endomorphisms, and a product system E over Γ^+. A *covariant representation* of (A, Γ^+, α, E) on a Hilbert space H is a pair (π, ϕ) where π is a unital representation of A and ϕ is a representation of E on H such that $\pi \circ \alpha_x = \gamma_x^\phi \circ \pi$ for $x \in \Gamma^+$. If $\{u_i : i = 1, 2, \cdots, n\}$ is an orthonormal basis for E_x then this is equivalent to

$$\pi(\alpha_x(a)) = \sum_{i=1}^{n} \phi_x(u_i)\pi(a)\phi_x(u_i)^* \text{ for } x \in \Gamma^+, a \in A. \tag{3.2}$$

A *crossed product* for (A, Γ^+, α, E) is a C^*-algebra $A \times_{\alpha,E} \Gamma^+$ together with a unital homomorphism $i_A : A \to A \times_{\alpha,E} \Gamma^+$ and a representation $i_E : E \to A \times_{\alpha,E} \Gamma^+$ satisfy

(1) if $a \in A$, $x \in \Gamma^+$ and $\{u_i : i = 1, 2, \cdots, n\}$ is an orthonormal basis for E_x, then $i_A(\alpha_x(a) = \sum_{i=1}^n i_E(u_i) i_A i_E(u_i)^*$.
(2) for every covariant representation (π, ϕ) of (A, Γ^+, α, E), there is a unital representation $\pi \times \phi$ of $A \times_{\alpha,E} \Gamma^+$ such that $(\pi \times \phi) \circ i_A = \pi$ and $(\pi \times \phi) \circ i_E = \phi$
(3) the C^*-algebra $A \times_{\alpha,E} \Gamma^+$ is generated by $i_A(A) \cup i_E(E)$.

Such crossed product always exists provided the system has a covariant representation, and it is unique up to isomorphism [5, Proposition 2.6].

Lemma 3.4. *If $E = (\Gamma^+ \times \mathbb{C})^\sigma$, then covariant representations of (A, Γ^+, α, E) are covariant representations of the system $(A, \Gamma^+, \alpha, \sigma)$. Therefore $A \times_{\alpha,E} \Gamma^+$ is the twisted semigroup crossed product $A \times_{\alpha,\sigma} \Gamma^+$.*

Proof. Let (π, ϕ) be a covariant representation of $(A, \Gamma^+, \alpha, (\Gamma^+ \times \mathbb{C})^\sigma)$. By Lemma 3.2, $x \in \Gamma^+ \mapsto \phi_x(1)$ is an isometric σ-representation of Γ^+. Each $E_x = \mathbb{C}$ has orthornormal basis $\{1\}$, hence the covariant condition in (3.2) is $\pi(\alpha_x(a)) = \phi_x(1)\pi(a)\phi_x(1)^*$ for all $a \in A$ and $x \in \Gamma^+$, which is the usual covariant property of a representation of $(A, \Gamma^+, \alpha, \sigma)$.

Consequently if (j_A, j_{Γ^+}) is the canonical covariant pair of $(A, \Gamma^+, \alpha, \sigma)$ in the crossed product $A \times_{\alpha,\sigma} \Gamma^+$, then $j_E : E \to B(H)$ defined by $j_x(u) = u j_{\Gamma^+}(x)$ for $u \in E_x$ is a representation of E. The pair (j_A, j_E) satisfies the following relation:

$$j_A(\alpha_x(a)) = j_{\Gamma^+}(x) j_A(a) j_{\Gamma^+}(x)^* = j_x(1) j_A(a) j_x(1)^* \text{ for } x \in \Gamma^+, a \in A.$$

So $(A \times_{\alpha,\sigma} \Gamma^+, j_A, j_E)$ is a crossed product for (A, Γ^+, α, E), and hence by the uniqueness property $A \times_{\alpha,(\Gamma^+\times\mathbb{C})^\sigma} \Gamma^+$ is isomorphic to the twisted crossed product $A \times_{\alpha,\sigma} \Gamma^+$. $\square$

We can see from this lemma that $B_{\Gamma^+} \times_{\tau,(\Gamma^+\times\mathbb{C})^\sigma} \Gamma^+$ is the twisted crossed product $B_{\Gamma^+} \times_{\tau,\sigma} \Gamma^+$. The Proposition 4.1 [5] proves that each representation ϕ of $E = (\Gamma^+ \times \mathbb{C})^\sigma$ induces a representation π_ϕ of B_{Γ^+} such that $\pi_\phi(1_x) = \gamma_x^\phi(I)$ and that the pair (π_ϕ, ϕ) is a covariant representation of $(B_{\Gamma^+}, \Gamma^+, \tau, E) = (\Gamma^+ \times \mathbb{C})^\sigma)$. And the representation π_ϕ is faithful iff $\Pi_{k=1}^n(I - \gamma_{x_k}^\phi(I)) \neq 0$ for every finite subset $\{x_1, x_2, \cdots, x_n\}$ of Γ^+ containing nonzero element.

Since we have seen that representations ϕ of $E = (\Gamma^+\times\mathbb{C})^\sigma$ correspond to isometric σ-representations V of Γ^+ given by $V_x = \phi_x(1)$, it follows that there is a pair of covariant representation (π_V, V) of $(B_{\Gamma^+}, \Gamma^+, \tau, \sigma)$ such that $\pi_V(1_x) = \gamma_x^\phi(I) = V_xV_x^*$, and that π_V is faithful iff $\Pi_{k=1}^n(I - V_{x_k}V_{x_k}^*) \neq 0$ for every finite subset $\{x_1, x_2, \cdots, x_n\}$ of Γ^+ containing nonzero element, which is equivalent to V_x is nonunitary for all x. So we now translate Proposition 4.1 and Theorem 5.1 of [5] into the next two propositions.

Proposition 3.5. (1) *For any isometric σ-representation V of Γ^+ on H, there is a representation π_V of B_{Γ^+} such that the pair (π_V, V) is a covariant representation of $(B_{\Gamma^+}, \Gamma^+, \tau, \sigma)$. If V_x is nonunitary for all $x \neq 0$, then π_V is faithful.*

(2) *$B_{\Gamma^+} \times_{\tau,\sigma} \Gamma^+$ is generated by $\{i_{\Gamma^+}(x) : x \in \Gamma^+\}$; it is the closure of*

$$\operatorname{span}\{i_{\Gamma^+}(x)i_{\Gamma^+}(y)^* : x, y \in \Gamma^+\}.$$

(3) *The homomorphism $i_{B_{\Gamma^+}} : B_{\Gamma^+} \to B_{\Gamma^+} \times_{\tau,\sigma} \Gamma^+$ is injective.*

Proposition 3.6. *If V is an isometric σ-representation of Γ^+, then $\pi_V \times V$ is an isomorphism of $B_{\Gamma^+} \times_{\tau,\sigma} \Gamma^+$ onto the C^*-algebra $C^*(V_x, \sigma)$ generated by isometric σ-representation of Γ^+ if and only if V is nonunitary.*

As a result the C^*-algebras generated by two isometric σ-representations V and W of nonunitary isometries are canonically isomorphic:

Corollary 3.7. *Let Γ be a totally ordered abelian group with its positive cone Γ^+, and let σ be a cocycle on Γ. If V and W are two nonunitary isometric σ-representations of Γ^+. Then the map $V_x \mapsto W_x$ extends to an isomorphism of $C^*(V_x, \sigma)$ onto $C^*(W_x, \sigma)$.*

Proof. The map $(\pi_V \times V)^{-1} \circ (\pi_W \times W)$ is, by Proposition 3.6, an isomorphism of the C^*-algebra $C^*(V_x : x \in \Gamma^+)$ onto $C^*(W_x : x \in \Gamma^+)$ such that V_x is mapped into W_x for all $x \in \Gamma^+$. $\square$

4. Generalization of Ji's theorem

We now recall the algebra studied by Ji in [6]. Suppose Γ^+ is a totally ordered abelian group with positive cone Γ^+, and σ is a cocycle in Γ satisfying $\sigma(x, -x) = 1$ for all x. For $x \in \Gamma^+$, let T_x be an isometry on $\ell^2(\Gamma^+)$ defined by

$$T_xf(y) = \begin{cases} \sigma(-y, x)f(y - x) & \text{for } y \geq x \\ 0 & \text{for } 0 \leq y < x. \end{cases} \tag{4.1}$$

Each T_x is a nonunitary isometry, and $T_xT_y = \sigma(x,y)T_{x+y}$ for all $x, y \in \Gamma^+$. Let $\mathcal{T}_\Gamma^\sigma$ be the C^*-algebra generated by $\{T_x : x \in \Gamma^+\}$, it is called the *twisted Toeplitz algebra.*

Remark 4.1. Notice that $T : x \mapsto T_x$ is a nonunitary isometric σ-representation of Γ^+, by Corollary 3.7 the twisted Toeplitz algebra $\mathcal{T}_\Gamma^\sigma$ is universal for isometric σ-representation of Γ^+. This generalizes Theorem 1 (1) of Ji [6] in the case Γ is a dense subgroup of real numbers. Part (ii) of his theorem can also be recovered from our results: $\mathcal{T}_\Gamma^\sigma$ is isomorphic to $B_{\Gamma^+} \times_{\tau,\sigma} \Gamma^+$, and which is independent of the choice of representative σ for the class σ by Lemma 2.4.

To obtain the exact sequence as in the part (iii) of Ji's theorem, we apply our exact sequence of twisted semigroup crossed products. Consider the ideal $B_{\Gamma^+,\infty} := \overline{\operatorname{span}}\{1_x - 1_y : x < y \text{ in } \Gamma^+\}$ of B_{Γ^+}. It was proved in [2] that $B_{\Gamma^+,\infty}$ is an extendible τ-invariant ideal of B_{Γ^+} (an approximate identity $(1_0 - 1_y)_{y\in\Gamma^+}$ satisfies $\tau_x((1_0 - 1_y)_{y\in\Gamma^+}) \longrightarrow 1_x = \tau_x(1_0)$ strictly in $M(B_{\Gamma^+,\infty})$). So Theorem 2.5 gives the following exact sequence:

$$0 \longrightarrow B_{\Gamma^+,\infty} \times_{\tau,\sigma} \Gamma^+ \xrightarrow{\Phi} B_{\Gamma^+} \times_{\tau,\sigma} \Gamma^+ \xrightarrow{\Psi} B_{\Gamma^+}/B_{\Gamma^+,\infty} \times_{\tilde{\tau},\sigma} \Gamma^+ \longrightarrow 0. \tag{4.2}$$

We claim that $B_{\Gamma^+}/B_{\Gamma^+,\infty} \times_{\tilde{\tau},\sigma} \Gamma^+$ is the twisted group C^*-algebra $C^*(\Gamma,\sigma)$ generated by unitary σ-representation of the group Γ. To see this, note that $\epsilon : f \in B_{\Gamma^+} \mapsto \lim_{x\in\infty} f(x) \in \mathbb{C}$ induces an isomorphism $\tilde{\epsilon}$ of $B_{\Gamma^+}/B_{\Gamma^+,\infty}$ onto $\mathbb{C}$ such that $\tilde{\epsilon}(\tau_z(f)) = \epsilon(f)$ for all $z \in \Gamma^+$ and $f \in B_{\Gamma^+}$. Therefore the two systems $(B_{\Gamma^+}/B_{\Gamma^+,\infty}, \Gamma^+, \tilde{\tau}, \sigma)$ and $(\mathbb{C}, \Gamma^+, \mathrm{id}, \sigma)$ are equivalent. Hence $B_{\Gamma^+}/B_{\Gamma^+,\infty} \times_{\tilde{\tau},\sigma} \Gamma^+$ is isomorphic to $\mathbb{C} \times_{\mathrm{id},\sigma} \Gamma^+$. Since covariant representations of $(\mathbb{C}, \Gamma^+, \mathrm{id}, \sigma)$ are in one to one correspondence to unitary σ-representations of the group Γ, it follows that $\mathbb{C} \times_{\mathrm{id},\sigma} \Gamma^+$ is the twisted group C^*-algebra $C^*(\Gamma,\sigma)$. Thus $B_{\Gamma^+}/B_{\Gamma^+,\infty} \times_{\tilde{\tau},\sigma} \Gamma^+$ is indeed isomorphic to $C^*(\Gamma,\sigma)$ as we claimed.

Lemma 4.2. *The isomorphism $\pi_T \times T$ of $B_{\Gamma^+} \times_{\tau,\sigma} \Gamma^+$ onto $\mathcal{T}_\Gamma^\sigma$ takes $B_{\Gamma^+,\infty} \times_{\tau,\sigma} \Gamma^+$ to the twisted commutator ideal $\mathcal{C}_\Gamma^\sigma$ of $\mathcal{T}_\Gamma^\sigma$ generated by the subset*

$$\{T_xT_y - \sigma(x,y)\overline{\sigma(y,x)}T_yT_x : x, y \in \Gamma\} \textit{ where } T_x := T_{-x}^* \textit{for } x < 0.$$

Proof. Note that the ideal $\Phi(B_{\Gamma^+,\infty} \times_{\tau,\sigma} \Gamma^+)$ in 4.2 is

$$D := \overline{\operatorname{span}}\{i_{\Gamma^+}(x)^* i_{B_{\Gamma^+}}(1_z - 1_w) i_{\Gamma^+}(y) : x, y, z, w \in \Gamma^+, z \leq w\}.$$

Since the isomorphism $\pi_T \times T$ takes $i_{B_{\Gamma^+}}(1_x)$ to $T_xT_x^*$ and $i_{\Gamma^+}(x)$ to T_x, it follows that the ideal $\pi_T \times T(D)$ is

$$\overline{\operatorname{span}}\{T_x^*(T_zT_z^* - T_wT_w^*)T_y : x, y, z, w \in \Gamma^+, z \leq w\}. \tag{4.3}$$

We want to show that (4.3) is the twisted commutator ideal $\mathcal{C}_\Gamma^\sigma$.

By the property of cocycle, we can see that $T_xT_y - \sigma(x,y)\overline{\sigma y, x}T_yT_x = 0$ when x and y are both positive, and also when x and y are both negative. So the ideal $\mathcal{C}_\Gamma^\sigma$ of $\mathcal{T}_\Gamma^\sigma$ is generated by

$$\{[T_xT_y]_\sigma = T_xT_y^* - \sigma(x,-y)\overline{\sigma(-y,x)}T_y^*T_x : x, y \geq 0\}.$$

We prove $\mathcal{C}_\Gamma^\sigma$ is contained in $\pi_T \times T(D)$. For this we show that $T_xT_y^*$ is equivalent to $\overline{\sigma(-y,x)}T_y^*T_x$ modulo $\pi_T \times T(D)$ for all $x, y \in \Gamma^+$. If $x \geq y$ in Γ^+, then $T_xT_y^* =$

$\overline{\sigma(x-y,y)}T_{x-y}T_yT_y^*$, which is equivalent to $\overline{\sigma(x-y,y)}T_{x-y}$ because $T_yT_y^*-1$ lies in $\pi_T\times T(D)$. By doing some computations involved only the property of cocycle, we see that

$$\begin{aligned}\overline{\sigma(x-y,y)}T_{x-y} &= \overline{\sigma(x-y,y)}T_y^*T_yT_{x-y} = \overline{\sigma(x-y,y)}\sigma(y,x-y)T_y^*T_x\\ &= \overline{\sigma(x-y,y)}\sigma(y,-y+x)T_y^*T_x\\ &= [\sigma(x,-y)\overline{\sigma(-y,y)\sigma(x,0)}][\sigma(y,-y)\sigma(0,x)\overline{\sigma(-y,x)}]T_y^*T_x\\ &= \sigma(x,-y)\overline{\sigma(-y,x)}T_y^*T_x.\end{aligned}$$

So $T_xT_y^*$ is equivalent to $\sigma(x,-y)\overline{\sigma(-y,x)}T_y^*T_x$ modulo $\pi_T\times T(D)$. Hence the generator $[T_xT_y]=T_xT_y^*-\sigma(x,-y)\overline{\sigma(-y,x)}T_y^*T_x$ belongs to $\pi_T\times T(D)$ for all $x\geq y$.

If $x<y$ in Γ^+ then from the previous case we know that $[T_yT_x]_\sigma$ is an element of $\pi_T\times T(D)$. Since our group is abelian, $(x,y)\mapsto\sigma(x,y)\overline{\sigma(y,x)}$ is a homomorphism in both variables, which implies $\sigma(y,-x)\overline{\sigma(-x,y)}=\sigma(-y,x)\overline{\sigma(x,-y)}$. Therefore $[T_y,T_x]_\sigma\in\pi_T\times T(D)$ if and only if $[T_x,T_y]_\sigma\in\pi_T\times T(D)$. Thus $\mathcal{C}_\Gamma^\sigma$ is contained in $\pi_T\times T(D)$.

Conversely, for any $z\leq w$ in Γ^+, we write $T_zT_z^*-T_wT_w^*=(T_zT_z^*-1)-(T_wT_w^*-1)$, and since $T_zT_z^*-1=T_zT_z^*-\sigma(-z,z)\overline{\sigma(z,-z)}T_z^*T_z$ is an element of $\mathcal{C}_\Gamma^\sigma$, it follows that $\pi_T\times T(D)$ is contained in $\mathcal{C}_\Gamma^\sigma$. □

So Theorem 1 (iii) of [6] is recovered from the next corollary.

Corollary 4.3. *Let Γ be an abelian totally ordered group, and σ a normalized two-cocycle of Γ into $\mathbb{T}$. Suppose $\mathcal{T}_\Gamma^\sigma$ is the twisted Toeplitz algebra, $\mathcal{C}_\Gamma^\sigma$ is the twisted commutator ideal of $\mathcal{T}_\Gamma^\sigma$, and $C^*(\Gamma,\sigma)$ is the twisted group C^*-algebra. Then there is a short exact sequence of C^*-algebras:*

$$0\longrightarrow\mathcal{C}_\Gamma^\sigma\longrightarrow\mathcal{T}_\Gamma^\sigma\longrightarrow C^*(\Gamma,\sigma)\longrightarrow 0.$$

Proof. It follows from Remark 4.1 and Lemma 4.2. □

Corollary 4.4. *If Γ is an Archimedean group, then $\mathcal{C}_\Gamma^\sigma$ is simple.*

Proof. Suppose $\mathcal{I}$ is a nonzero ideal in $\mathcal{C}_\Gamma^\sigma\simeq B_{\Gamma^+,\infty}\times_{\tau,\sigma}\Gamma^+$. Let π be a faithful representation of $B_{\Gamma^+}\times_{\tau,\sigma}\Gamma^+$ such that $\ker\pi=\mathcal{I}$. An application of Proposition 3.5 shows that $\pi=\rho_W\times W$ for an isometric σ-representation W of Γ^+. Since $\pi=\rho_W\times W$ is not faithful, by Proposition 3.6 there exists a nonzero $x\in\Gamma^+$ such that $W_xW_x^*=1$. For $y\in\Gamma^+$, the Archimedean hypothesis implies $0<y<nx$ for some $n\in\mathbb{N}$ Notice that $I-W_yW_y^*\leq I-W_{nx}W_{nx}^*$ and $W_{nx}W_{nx}^*=1$, the isometry W_y must be unitary for all $y\in\Gamma^+$. So the representation ρ_W vanishes on $B_{\Gamma^+,\infty}$ because $\rho_W(1_s-1_t)=W_sW_s^*-W_tW_t^*=0$ for $s\leq t\in\Gamma^+$. Therefore we have $\rho_W\times W=0$ on $B_{\Gamma^+,\infty}\times_{\tau,\sigma}\Gamma^+$. Thus $B_{\Gamma^+,\infty}\times_{\tau,\sigma}\Gamma^+\simeq\mathcal{C}_\Gamma^\sigma$ is contained in $\mathcal{I}$, hence $\mathcal{C}_\Gamma^\sigma$ is simple. □

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