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PRIMITIVE IDEALS OF TOEPLITZ ALGEBRAS OF ORDERED GROUPS

Rizky Rosjanuardi

Abstract. The topology on primitive ideal space of Toeplitz algebras of totally ordered abelian groups can be identified through the upwards-looking topology if and only if the chain of order ideals is well-ordered. We describe the topology on primitive ideal space of Toeplitz algebra of totally ordered abelian groups when the chain of order ideals is not well ordered.

1 Introduction

Investigation on ideals structure of C^*-algebras is always an interesting topic. This is motivated by investigations on ideal structure of commutative C^*-algebra which turns out to be a very beautiful conclusion. Gelfand and Naimark proved that there is only one (up to isomorphisms) commutative C^*-algebra, i.e. the algebra $C(\Delta)$ consists of all continuous functions on a maximal ideal Δ. Attempts to make analogies of theory of Gelfand and Naimark are highly depend on particular algebras, and have led ones to investigate ideals structure in general and primitive ideals, for example in the case of Toepltz algebra, see [2, 13, 4].

Suppose Γ^+ is the positive cone of a totally ordered abelian group Γ. We now recall the notion of Toeplitz algebra. Denote by e_x the characteristic function of the singleton set $\{x\}$. The set $\{e_x : x \in \Gamma^+\}$ is then the usual basis for the Hilbert space $\ell^2(\Gamma^+)$. For every $x \in \Gamma^+$, denote by T_x the isometric linear operator on $\ell^2(\Gamma^+)$ which satisfies $T_x(e_y) = e_{y+x}$ for every $y \in \Gamma^+$. The Toeplitz algebra of Γ is the universal C^*-subalgebra $\mathcal{T}(\Gamma)$ of $B(\ell^2(\Gamma^+))$ generated by operators $\{T_x : x \in \Gamma^+\}$.

In [2] Adji and Raeburn investigate primitive ideals of Toeplitz algebra $\mathcal{T}(\Gamma)$ of totally ordered abelian group Γ. This covers groups which

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are not finitely generated, which is more general than those considered by Murphy [10]. Adji and Raeburn [2] show that each irreducible representation of the Toeplitz algebra $\mathcal{T}(\Gamma)$ factors through an irreducible representation of $\mathcal{T}(\Gamma/I)$ for some $I \in \Sigma(\Gamma)$, and that there is a bijection map L of the disjoint union $X := \bigsqcup\{\hat{I} : I \in \Sigma(\Gamma)\}$ onto the set $\operatorname{Prim}\mathcal{T}(\Gamma)$ of primitive ideals of $\mathcal{T}(\Gamma)$. Further they show that under certain hypotheses, the map L is a homeomorphism. Later Adji, Raeburn and Rosjanuardi [13, 4] showed that the homeomorphism could be established under weaker hypotheses. The hypothesis is on the chain of order ideals $\Sigma(\Gamma)$ of totally ordered abelian groups Γ. In [13, 4], the hypothesis on $\Sigma(\Gamma)$ is well ordered, which is weaker than isomorphis to a subset of $\mathbb{N} \cup \{\infty\}$ as required in [2].

As it is pointed out in [4, Remark 4.3], that in general, though, the structure of $\Sigma(\Gamma)$ could be more complicated than that $\Sigma(\Gamma)$ is well ordered. In this paper we figure out the topology on $\operatorname{Prim}\mathcal{T}(\Gamma)$ for more general Γ. Our main results are basically applications of theorems in [4].

2 Order Ideals in a Totally Ordered Abelian Group

Suppose $(\Gamma, +)$ is a discrete abelian group. A **partial ordering** on Γ is a binary relation $\geq$ on Γ such that for every $x, y, z \in G$ satisfy these properties:

(i) $x \geq x$ (reflexive).

(ii) If $x \geq y$ and $y \geq x$ then $x = y$ (antisymmetric).

(iii) If $x \geq y$ and $y \geq z$ then $x \geq z$ (transitive).

If every 2 elements of Γ are comparable i.e $x \leq y$ or $y \leq x$ for every $x, y \in \Gamma$, we call the relation $\leq$ is a total ordering on Γ. The group Γ is then called a **totally ordered discrete abelian group** or just **ordered group** when the context is clear. The **positive cone** Γ^+ of Γ is $\{x \in \Gamma : x \geq 0\}$, it is a semigroup, because it inherits all properties of group Γ, except the existence of inverse.

Suppose $(\Gamma_1, \leq_1)$, $(\Gamma_2, \leq_2)$ are totally ordered abelian groups. A group homomorphism $\varphi : \Gamma_1 \longrightarrow \Gamma_2$ is called an **order homomorphism** if it is order preserving, i.e if $x \leq_1 y$ implies $\varphi(x) \leq_2 \varphi(y)$ for every $x, y \in \Gamma_1$. In other language we have $\varphi(\Gamma_1^+) \subseteq \Gamma_2^+$.

An **order ideal** of a totally ordered abelian group Γ is a subgroup I which is order preserving, in the sense that if $x \in \Gamma^+, y \in I^+$ with $x \leq y$ then $x \in I$. The set $\Sigma(\Gamma)$ of order ideals is totally ordered by inclusion. To see this, let $I, J \in \Sigma(\Gamma)$. Suppose there is an element $y \in J^+ - I$. This forces $0 < x \leq y$ for all $x \in I^+$ (because if there is $x_0 \in I^+$ such that $0 < y < x_0$ then $y \in I$ which is a contradiction), thus $I^+ \subseteq J^+$. Therefore $I \subseteq J$. In case $J^+ - I = \emptyset$, it means that $J^+ \subseteq I^+$. Thus $J \subseteq I$.

The **quotient group** Γ/I is defined in the similar manner of usual quotient group, and there is a quotient order on Γ/I defined by

$$xI \leq yI \Leftrightarrow \exists i \in I \text{ such that } x \leq y + i,$$

this is also a total order. Moreover, if $q : \Gamma \longrightarrow \Gamma/I$ is the quotient homomorphism then $q(\Gamma^+) = (\Gamma/I)^+$.

A totally ordered abelian group $(\Gamma, \leq)$ is called **Archimedean** if for every $x, y \in \Gamma^+$ there is $n \in \mathbb{N}$ such that $y \leq nx$. Fuchs in [6][Theorem 1, p.47] verified that a totally ordered Γ is Archimedean if and only if it is simple (its only order ideals are 0 and Γ).

Lemma 1. *Suppose I is a nontrivial order ideal of a totally ordered abelian group Γ. Then every nontrivial ideal of Γ/I is of the form J/I where $J \in \Sigma(\Gamma)$, and $J \supsetneq I$.*

Proof. Suppose the contrary. If $J \subseteqq I$, then J/I is a trivial ideal which is a contradiction. If $J \notin \Sigma(\Gamma)$, then there is $y \in J^+$ and $x \in \Gamma$ such that $0 < x \leq y$, but $x \notin J$. The quotient order implies that

$$I < x + I \leq y + I.$$

But then $x + I \notin J/I$, because $x \notin J$. This implies that J/I is not an order ideal, and this contradicts the fact that $J/I \in \Sigma(\Gamma/I)$. □

Lemma 2. *Let I be an order ideal of a totally ordered abelian group Γ. Suppose J_1 and J_2 are ideals such that $J_1/I \subseteq J_2/I$. Then $J_1 \subseteq J_2$.*

Proof. Suppose the contrary; $J_1 \nsubseteq J_2$. Then there is an element x of J_1 which is not in J_2. This implies that $x + I \in J_1/I$, but $x + I \notin J_2/I$. This is impossible, because $J_1/I \subseteq J_2/I$. □

3 Primitive Ideals of Toeplitz Algebras

Adji and Raeburn in [2] study the ideal structure of the Toeplitz algebras of a totally ordered abelian groups. We shall give an overview of what they have done. Suppose Γ is a totally ordered abelian group. The set $\Sigma(\Gamma)$ of order ideals is a totally ordered by inclusion. They show in Theorem 3.1 that each irreducible representation of the Toeplitz algebra $\mathcal{T}(\Gamma)$ factors through an irreducible representation of $\mathcal{T}(\Gamma/I)$ for some $I \in \Sigma(\Gamma)$, and that there is a bijection map L of the disjoint union $X := \bigsqcup\{\hat{I} : I \in \Sigma(\Gamma)\}$ onto the set $\operatorname{Prim}\mathcal{T}(\Gamma)$ of primitive ideals of $\mathcal{T}(\Gamma)$. To see how L is defined, let I be an order ideal of Γ. Then the map $x \longmapsto T^{\Gamma/I}_{x+I}$ is an isometric representation of Γ^+ in $\mathcal{T}(\Gamma/I)$. Therefore by the universality of $\mathcal{T}(\Gamma)$, there is a homomorphism $Q_I : \mathcal{T}(\Gamma) \longrightarrow \mathcal{T}(\Gamma/I)$ such that $Q_I(T^\Gamma_x) = T^{\Gamma/I}_{x+I}$, and that Q_I is surjective. Suppose $\mathcal{C}(\Gamma, I)$ denotes the ideal in $\mathcal{T}(\Gamma)$ generated by $\{T_u T_u^* - T_v T_v^* : v - u \in I^+\}$. It is proved in [12, Theorem 3.1] that there is a short exact sequence of C^*-algebras:

$$0 \to \mathcal{C}(\Gamma, I) \to \mathcal{T}(\Gamma) \xrightarrow{\phi_I} \operatorname{Ind}^{\hat{\Gamma}}_{I^\perp}(\mathcal{T}(\Gamma/I), \alpha^{\Gamma/I}) \to 0. \tag{3.1}$$

in which $\phi_I(a)(\gamma) = Q_I \circ (\alpha^\Gamma_\gamma)^{-1}(a)$ for $a \in \mathcal{T}(\Gamma)$, $\gamma \in \hat{\Gamma}$, and α^Γ_γ is dual action of $\hat{\Gamma}$ on $\mathcal{T}(\Gamma)$ characterised by $\alpha^\Gamma_\gamma(T_x) = \gamma(x)T_x$. The identity representation $T^{\Gamma/I}$ of $\mathcal{T}(\Gamma/I)$ is irreducible [9], it follows from [14, Proposition 6.16] that $\ker Q_I \circ (\alpha^\Gamma_\gamma)^{-1}$ is a primitive ideal of $\mathcal{T}(\Gamma)$. Moreover since

$$Q_I \circ \alpha^\Gamma_\chi = \alpha^{\Gamma/I}_\chi \circ Q_I \text{ for } \chi \in I^\perp = \hat{\Gamma/I}, \tag{3.2}$$

the map $\gamma \longmapsto \ker Q_I \circ \alpha_\gamma^{-1}$ is constant on $I^\perp$ cosets in $\hat{\Gamma}$. Therefore it induces a well defined map L of $\hat{I} = \hat{\Gamma}/I^\perp$ into $\operatorname{Prim}\mathcal{T}(\Gamma)$. They write $L(I, \gamma)$ for the image of $\gamma \in \hat{I}$, so that

$$L(I, \gamma) := \ker Q_I \circ {\alpha^\Gamma_\nu}^{-1} \text{ where } \nu \in \hat{\Gamma} \text{ satisfies } \nu|_I = \gamma, \tag{3.3}$$

and then they prove in Theorem 3.1 that L is a bijection.

Using the bijection $L : (I, \gamma) \in X \longmapsto \ker Q_I \circ {\alpha^\Gamma_\nu}^{-1} \in \operatorname{Prim}\mathcal{T}(\Gamma)$, Adji and Raeburn describe a new topology on X which corresponds to the hull-kernel topology on $\operatorname{Prim}\mathcal{T}(\Gamma)$. This new tology, is later called the *upwards-looking* topology. They topologise X by specifying the closure operation as stated in the following definition.

Definition 3. [2] The closure $\overline{F}$ of a subset F of X is the set consisting of all pairs (J, γ) where J is an order ideal and $\gamma \in \hat{J}$ such that for every open neighbourhood N of γ in $\hat{J}$, there exists $I \in \Sigma(\Gamma)$ and $\chi \in N$ for which $I \subset J$ and $(I, \chi|_I) \in F$.

They prove that this is the hull-kernel topology of $\operatorname{Prim}\mathcal{T}(\Gamma)$ when Γ is a group such that the set $\Sigma(\Gamma)$ of order ideal is order isomorphic to a subset of $\mathbb{N} \cup \{\infty\}$. The proof rely on the case-by-case analysis depending on the set $\Sigma(\Gamma)$ of order ideals of Γ. Beginning with the groups Γ where $\Sigma(\Gamma)$ is finite, this includes all finitely generated ordered groups (hence lexicographic finite direct sum of Archimedean ordered groups), they show in Corollary 4.6 the map L is a homeomorphism. It is pointed out in Example 2.1 that the hypothesis finite $\Sigma(\Gamma)$ is in fact more general than lexicographic direct sum: there is a group Γ which $\Sigma(\Gamma)$ is finite but Γ is not lexicographic direct sum. They then considered the case where $\Sigma(\Gamma)$ is infinite. It is proved in Proposition 4.7 that if $\Sigma(\Gamma)$ is isomorphic to a subset of $\mathbb{N} \cup \{+\infty\}$, the topology of $\operatorname{Prim} \mathcal{T}(\Gamma)$ can be described by the topology on X. However beyond the hypothesis, there are found problems, for examples: if $\Gamma =$ antilexicographic direct sums over $\mathbb{N}$ they could have $\Gamma = \bigcup_{n \in \mathbb{N}} I_n$, or if $\Gamma =$ lexicographic direct sums over $\mathbb{N}$ then $\bigcap_{n \in \mathbb{N}} I_n$ could be $\{0\}$. It is given in Theorem 4.13 that if $\Sigma(\Gamma)$ is isomorphic to a subset of $\{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}$, which will include all the two examples, then the topology of $\operatorname{Prim} \mathcal{T}(\Gamma)$ can be described by firstly amending the topology they defined on X.

In [4], Raeburn and his collaborators extended the results in [2]. Their main theorem, says that $\operatorname{Prim} \mathcal{T}(\Gamma)$ is homeomorphic to $X(\Gamma)$ with the upwards-looking topology if and only if the totally ordered set $\Sigma(\Gamma)$ is well-ordered in the sense that every non-empty subset has a least element. They uses the structure theorem, and a new general result on the upwards-looking topology ([4, Proposition 3.3]). The proof of Proposition 3.3 uses classical Toeplitz operators as well as the universal property of $\mathcal{T}(\Gamma)$ which was the main tool in [2]. Then they described $\operatorname{Prim} \mathcal{T}(\Gamma)$ when parts of $\Sigma(\Gamma)$ are well-ordered.

Our interest is describe the primitive ideal spaces $\mathcal{T}(\Gamma)$ when $\Sigma(\Gamma)$ is not well ordered. To prepare for the proofs of our results, we recall from [4] some general results relating the upwards-looking topology to the topology on $\operatorname{Prim}(\mathcal{T}(\Gamma)$ when parts of $\Sigma(\Gamma)$ are well ordered.

Theorem 4. *Let Γ be a totally ordered abelian group, and denote by $X(\Gamma)$ the disjoint union $\bigsqcup\{\widehat{I} : I \in \Sigma(\Gamma)\}$. The map $L : X(\Gamma) \to \operatorname{Prim} \mathcal{T}(\Gamma)$ is a homeomorphism for the upwards-looking topology on $X(\Gamma)$ if and only if $\Sigma(\Gamma)$ is well-ordered.*

Proof. This is proved in Theorem 3.1 of [4] □

Theorem 5. *Let Γ be a totally ordered abelian group and I an order ideal in Γ. Suppose that $\Sigma(\Gamma/I)$ is well-ordered, and give the set*

$$X(\Gamma, I) := \bigsqcup\{\widehat{J} : J \in \Sigma(\Gamma),\ I \subset J\}$$

the upwards-looking topology. Then $L : X(\Gamma, I) \to \operatorname{Prim} \mathcal{T}(\Gamma)$ is a homeomorphism of $X(\Gamma, I)$ onto the closed subset $\{P \in \operatorname{Prim} \mathcal{T}(\Gamma) : \mathcal{C}(\Gamma, I) \subset P\}$.

Proof. This is proved in Theorem 4.1 (b) of [4]. □

Proposition 6. *Suppose that* Γ *is a totally ordered abelian group such that the chain* $\Sigma(\Gamma)$ *of order ideals in* Γ *is isomorphic to a subset of* $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$. *For any* $I \in \Sigma(\Gamma)$, *the mapping* $L^{\Gamma/I}$ *is a homeomorphism of* $X(\Gamma/I)$ *onto* $\operatorname{Prim} \mathcal{T}(\Gamma/I)$.

Proof. Let $I \in \Sigma(\Gamma)$. Lemma 1 implies that the chain of order ideals in Γ/I is

$$I \subset J_1/I \subset J_2/I \subset ...,$$

where $J_i \in \Sigma(\Gamma)$ and $I \subset J_i \subset J_{i+1}$. Hence $\Sigma(\Gamma/I)$ is well ordered, and hence $L^{\Gamma/I}$ is a homeomorphism by Theorem 4.

□

Theorem 7. *Suppose that* Γ *is a totally ordered abelian group such that the chain* $\Sigma(\Gamma)$ *of order ideals in* Γ *is isomorphic to a subset of* $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$. *Then for any nonzero ideal* $I \in \Sigma(\Gamma)$, *the mapping* L^I *gives a homeomorphism of* $\sqcup\{\hat{J} : J \supset I\}$ *onto* $\{P \in \operatorname{Prim} \mathcal{T}(\Gamma) : \mathcal{C}(\Gamma, I) \subset P\}$.

Proof. From Proposition 6, $L^{\Gamma/I}$ is a homemomorphism of $X(\Gamma/I)$ onto $\operatorname{Prim} \mathcal{T}(\Gamma/I)$. Hence L^I is a homemomorphism by Theorem 5 □

Proposition 8. *Suppose that* Γ *is a totally ordered abelian group, and let* I *be an order ideal in* Γ *such that every oreder ideal* $J \supseteq I$ *has a successor. Then the mapping* $L^{\Gamma/I}$ *is a homeomorphism of* $X(\Gamma/I)$ *onto* $\operatorname{Prim} \mathcal{T}(\Gamma/I)$.

Proof. Let $I \in \Sigma(\Gamma)$ such that every order ideal $J \supseteq I$ has a successor. Since each nontrivial element of $\Sigma(\Gamma/I)$ is of the form J/I for $J \in \Sigma(\Gamma)$ and $J \supsetneqq I$, every element of $\Sigma(\Gamma/I)$ has a successor. This implies that $\Sigma(\Gamma/I)$ is well ordered. Hence $L^{\Gamma/I}$ is a homeomorphism by Theorem 4. □

Theorem 9. *Suppose that* Γ *is a totally ordered abelian group, and* $I \in \Sigma(\Gamma)$ *such that* $\Sigma(\Gamma/I) \cong \{-\infty\} \cup \mathbb{Z}\{\infty\}$. *Let* $J \in \Sigma(\Gamma)$ *such that* $J \supsetneqq I$. *Then* $\Sigma(\Gamma/J)$ *is well ordered and hence* $L^{\Gamma/J}$ *is a homeomorphism of* $X(\Gamma/J)$ *onto* $\operatorname{Prim} \mathcal{T}(\Gamma/J)$. *Hence* $L : (K, \chi) \mapsto L(K, \chi)$ *is a homeomorphism of* $\sqcup\{\hat{K} | K \in \Sigma(\Gamma) : K \supset J\}$ *onto* $\{P \in \operatorname{Prim} \mathcal{T}(\Gamma) : P \supset \mathcal{C}(\Gamma, J)\}$.

Proof. Lemma 1 and Lemma 2 say that we may write

$$\Sigma(\Gamma/I) := \{I = J_{-\infty} \subseteq ... \subseteq J_k/I \subseteq J_{k+1}/I \subseteq ... \subseteq \Gamma = J_\infty\}.$$

Now consider the subset

$$\mathcal{I} := \{I = J_{-\infty} \subseteq ... \subseteq J_k \subseteq J_{k+1} \subseteq ... \subseteq J_\infty = \Gamma\}$$

of $\Sigma(\Gamma)$. If $J \neq I$ be an element of I, i.e $J \in \Sigma(\Gamma)$ such that $J \supsetneqq I$, the set

$$\Sigma(\Gamma/J) = \{J \subseteq K_1/J \subseteq K_2/J...\}$$

is well ordered. Hence $L^{\Gamma/J}$ is a homeomorphism of $X(\Gamma/J)$ onto $\operatorname{Prim} \mathcal{T}(\Gamma/J)$ by Theorem 4. The second assertion is guaranteed by Theorem 5. □

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References

[1] S. Adji, M. Laca, M. Nilsen and I. Raeburn, *Crossed products by semigroups of endomorphisms and the Toeplitz algebras of ordered groups*, Proc. Amer. Math. Soc. **122**, Number 4 (1994), 1133-1141.

[2] S. Adji and I. Raeburn, *The ideal structure of Toeplitz algebra*, Integral Equation Operation Theory 48 (2004), 281-293.

[3] S. Adji and R. Rosjanuardi *Twisted semigroup crossed products and the twisted Toeplitz algebras of ordered groups*, submitted to Acta Mathematica Sinica.

[4] S. Adji, I. Raeburn and R. Rosjanuardi *Group extensions and the primitive ideal spaces of Toeplitz algebras*, Glasgow Math. Journal **49** (2007), 81-92.

[5] R. G. Douglas, *On the C^*-algebra of a one parameter semigroup of isometries*, Acta Math. **128** (1972), 143-151.

[6] L. Fuchs, *Partially ordered algebraic systems*, Pergamon Press, New York (1963)

[7] R. Ji, *Toeplitz operators on noncommutative tori and their real valued index*, Proc. of symposia in pure mathematics **51** part 2 (1990).

[8] A. Kumjian, D. Pask, I. Raeburn, J. Renault, *Graphs, Groupoids and Cuntz-Krieger Algebras*, J. Funct. Anal. 144 (1997) no. 2.

[9] G.J. Murphy, *Ordered group and Toeplitz algebras*, J. Operator Theory **18** (1987), 303-326.

[10] G.J. Murphy, *Toeplitz operators and algebras*, Math. Zeith. **208** (1991), 355-362.

[11] I. Raeburn, *On crossed products and Takai duality*, Proc. Edinburgh Math. Soc. **108** (1988), 321-330.

[12] S. Adji, *Semigroup crossed products and the structure of Toeplitz algebras*, J. Op. Theory **44** (2000), 139–150.

[13] R. Rosjanuardi, *Structure theory of twisted Toeplitz algebras and ideal structure of the (untwisted) Toeplitz algebras of ordered groups*, Dissertation, ITB, Indonesia (2005).

[14] D.P. William and I. Raeburn, *Morita equivalence and continuous-trace C^*-algebras*, Amer. Math. Soc., Providence, RI (1998).

Rizky Rosjanuardi: Department of Mathematics Education, Universitas Pendidikan Indonesia (UPI), Bandung 40154, Indonesia.
E-mail: rizky@upi.edu