Metric Spaces

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Abstract

This article covers almost all the concepts in metric spaces theory.

1 Introduction

Definition. A *metric space* is a set X together with a function $\rho : X \times X \to \mathbf{R}$ (called the *metric* of X), which satisfies the following properties for all $x, y, z \in X$:

M1. $\rho(x,y) \geq 0$ with $\rho(x,y) = 0$ if and only if $x = y$. (**Positive Definite**)

M2. $\rho(x,y) = \rho(y,x)$ (**Symmetric**)

M3. $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ (**Triangle Inequality**)

The pair (X, ρ) is then called metric space.

Example. Consider the set of real numbers $\mathbf{R}$ together with metric

$$\rho(x,y) = |x - y|.$$

All conditions of metric above follow directly from the properties of absolute value.

Example. Let X be a nonempty set. For $x, y \in X$ we define

$$\rho(x,y) = \left\{ \begin{array}{ll} 0 & , x = y \\ 1 & , x \neq y. \end{array} \right\}$$

This is what the so-called discrete metric.

Example. Consider the set $\mathbf{R}^n = \{(x_1, ..., x_n)\}$ together with

$$\rho(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}.$$

The first two properties are clear. We shall prove the third one, namely the triangle inequality.

Proof Let $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in \mathbf{R}^n$. Consider the function

$$\psi(u) = \sum_{i=1}^{n} (a_i u + b_i)^2, \quad u \in \mathbf{R}.$$

We get

$$\psi(u) = \sum_{i=1}^{n} a_i^2 u^2 + 2 \sum_{i=1}^{n} a_i b_i u + \sum_{i=1}^{n} b_i^2.$$

Since $\psi(u) \geq 0$, we have $(2 \sum_{i=1}^{n} a_i b_i)^2 - 4 \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq 0$. This implies the following inequality (Cauchy-Schwarz Inequality)

$$(\sum_{i=1}^{n} a_i b_i)^2 \leq \sum_{i=1}^{n} a_i^2 \ \sum_{i=1}^{n} b_i^2.$$

Accordingly, based on this inequality, we have

$$\sum_{i=1}^{n} a_i b_i \leq |\sum_{i=1}^{n} a_i b_i| \leq \sqrt{\sum_{i=1}^{n} a_i^2} \ \sqrt{\sum_{i=1}^{n} b_i^2}.$$

Finally, we get

$$\sum_{i=1}^{n} a_i^2 + 2 \sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2 \leq \sum_{i=1}^{n} a_i^2 + 2 \sqrt{\sum_{i=1}^{n} a_i^2} \ \sqrt{\sum_{i=1}^{n} b_i^2} + \sum_{i=1}^{n} b_i^2.$$

This inequality is equivalent to

$$\sum_{i=1}^{n} (a_i + b_i)^2 \leq \left(\sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2} \right)^2,$$

or

$$\sqrt{\sum_{i=1}^{n} (a_i + b_i)^2} \leq \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}.$$

If we set $a_i = x_i - z_i, \ b_i = z_i - y_i$ in this last form, we have the triangle inequality for the Euclidean metric.

Example. Let $\mathcal{C}[a,b]$ represent the collection of continuous function $f : [a,b] \to \mathbf{R}$ and

$$\|f\| = \sup_{x \in [a,b]} |f(x)|.$$

Then $\rho(f,g) = \|f - g\|$ is a metric on $\mathcal{C}[a,b]$. (**Prove!**)

Definition. Let $a \in X$ and $r > 0$. The *open ball* with *center* a and *radius* r is the set

$$B_r(a) = \{x \in X : \rho(x,a) < r\},$$

and the *closed ball* with *center* a and *radius* r is the set

$$\overline{B_r(a)} = \{x \in X : \rho(x,a) \leq r\}.$$

Definition.

i. A set $V \subset X$ is said to be open if for every $x \in V$ there is an $\varepsilon > 0$ such that the open ball $B_\varepsilon(x)$ is contained in V.

ii. A set $E \subset X$ is said to be *closed* if $E^c = X \setminus E$ is open.

Remark.

1. Every open ball is open, and every closed ball is closed.
2. If $a \in X$, then $X \setminus \{a\}$ is open and $\{a\}$ is closed.
3. In an arbitrary metric space, the empty set $\emptyset$ and the whole space X are both open and closed.
4. In the discrete space $\mathbf{R}$, every set is both open and closed.

Definition Let (x_n) be a sequence in a metric space X.

i. (x_n) *converges* (in X) if there is a point $a \in X$ (called the *limit*) such that for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$n \geq N \quad \text{implies} \quad \rho(x_n, a) < \varepsilon$$

ii. (x_n) is *Cauchy* if for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$n, m \geq N \quad \text{implies} \quad \rho(x_n, x_m) < \varepsilon$$

iii. (x_n) is bounded if there is an $M > 0$ and a $b \in X$ such that $\rho(x_n, b) \leq M$ for all $n \in \mathbf{N}$.

Moreover, we can establish the following result.
Theorem. Let X be a metric space.

i. A sequence in X can have at most one limit.

ii. If $x_n \in X$ converges to a and (x_{nk}) is any subsequence of (x_n) then x_{nk} converges to a as $k \to \infty$

iii. Every convergent sequence in X is bounded

iv. Every convergent sequence in X is Cauchy.

Remark. Let $x_n \in X$. Then $x_n \to a$ as $n \to \infty$ if and only if for every open set V which contains a there is an $N \in \mathbf{N}$ such that $n \geq N$ implies $x_n \in V$.

Theorem. Let $E \subset X$. Then E is closed if and only if the limit every convergent sequence $x_k \in E$ satisfies

$$\lim_{k \to \infty} x_k \in E.$$

Remark.

1. The discrete space contains bounded sequences which have no convergent subsequences.

2. The metric space $X = \mathbf{Q}$ contains Cauchy sequences which do not converge.