KALKULUS VARIASI

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TUJUAN

Mencari titik yang meminimumkan/memaksimumkan suatu fungsional

TOOL

Turunan Berarah

DIMENSI HINGGA

- **1 DIMENSI**

Algoritma Fermat ‘Jika f fungsi skalar 1 variabel yang terdiferensialkan f : R → R memuat nilai ekstrim di titik $\hat{x}$ maka f’($\hat{x}$) = 0’

- **N DIMENSI**

Didefinisikan F : $R^n \rightarrow R$
Turunan fungsi yang didefinisikan di R^n

$$\frac{d}{d\varepsilon} F(x+\varepsilon\eta)\,\Big|_{\varepsilon=0} = F'(x)\cdot\eta = \nabla F(x)\cdot\eta$$

Untuk x yang meminimumkan fungsi F di R^n maka

$$\nabla F(x)\cdot\eta = 0 \quad ,\forall\,\eta$$

Diperoleh

$$\nabla F(x)=0$$

$$\mathcal{L} : u \longrightarrow \mathcal{L}(u)$$

- admisable variation: $\eta \in T_u M$
  Memenuhi $x + \varepsilon\eta \in \mathrm{M}$

- **Turunan berarah:**

$$\delta\mathcal{L}(u;\eta; = \frac{d}{d\varepsilon}\mathcal{L}(u+\varepsilon\eta)\Big|_{\varepsilon=0} = \langle \delta\mathcal{L}(u),\eta \rangle + BCs$$

$$BCs = 0$$

$$\langle \delta\mathcal{L}(u),\eta \rangle = 0 \quad , \forall \eta$$

$$\delta\mathcal{L}(u) = 0 \quad \text{(Persamaan Euler - Lagrange)}$$

Contoh

$$\mathcal{L}(u) = \int_0^1 \tfrac{1}{2}(\partial_x u)^2 + u^2 dx$$

$$\mathcal{M} = \{u(x) | x \in [0,1], u(0) = 2\}$$

$$\mathcal{L} : \mathcal{M} \to R$$

Cari titik kritis dari $\mathcal{L}(u)$!

Penyelesaian

$$T_u\mathcal{M} = \{\eta(x) | x \in [0,1], \eta(0) = 0\}$$

$$\frac{d}{d\varepsilon}\mathcal{L}(u+\varepsilon\eta)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_0^1 \tfrac{1}{2}\left[\partial_x(u+\varepsilon\eta)^2 + (u+\varepsilon\eta)^2\right]dx\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\int_0^1 \tfrac{1}{2}\left[(\partial_x u + \varepsilon\partial_x\eta)^2 + (u^2 + 2\varepsilon u\eta + \varepsilon^2\eta^2)\right]dx\Big|_{\varepsilon=0}$$

$$= \int_0^1 \partial_x u \partial_x \eta dx + \int_0^1 2u\eta dx$$

$$= \partial_x u\eta\Big|_0^1 - \int_0^1 \partial_{xx} u\eta dx + \int_0^1 2u\eta dx$$

$$= (\partial_x u\eta)(1) - (\partial_x u\eta)(0) + \int_0^1 (-\partial_{xx}u + 2u)\eta dx$$

$$= (\partial_x u)(1)\eta(1) + \int_0^1 (-\partial_{xx}u + 2u)\eta dx$$

Titik kritis u diperoleh jika :

$$\frac{d}{d\varepsilon}\mathcal{L}(u+\varepsilon\eta)\Big|_{\varepsilon=0}=0$$

Sehingga diperoleh

$$(\partial_x u)(1)=0 \ \text{(Syarat batas natural)}$$

$$\int_0^1 (-\partial_{xx}u+2u)\eta\, dx=0 \Leftrightarrow \langle \delta\mathcal{L}(u),\eta\rangle = 0$$

Diperoleh Persamaan Euler – Lagrange(Turunan Variasi Pertama)

$$\delta\mathcal{L}(u)=0 \Leftrightarrow -\partial_{xx}u+2u=0$$

dengan syarat awal :

$$(\partial_x u)(1)=0, u(0)=2$$

$$u(x)=C_1e^{\sqrt{2}x}+C_2e^{-\sqrt{2}x} \ \text{dengan}\ C_1=\frac{2e^{-\sqrt{2}}}{e^{\sqrt{2}}+e^{-\sqrt{2}}} \ \text{dan}\ C_2=\frac{2e^{\sqrt{2}}}{e^{\sqrt{2}}+e^{-\sqrt{2}}}$$

Contoh

$$\mathcal{L}(u) = \int_I L(u,\dot{u},t)dt$$

$$\mathcal{L} : \mathcal{M} \to \mathrm{R}$$

Cari u yang merupakan titik kritis dari $\mathcal{L}$

Penyelesaian :

$$\frac{d}{d\varepsilon}\mathcal{L}(u+\varepsilon\eta)\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_I L(u+\varepsilon\eta,\dot{u}+\varepsilon\dot{\eta},t)dt\bigg|_{\varepsilon=0}$$

$$= \int_I \left(\frac{\partial L}{\partial u}\eta + \frac{\partial L}{\partial \dot{u}}\dot{\eta}\right)dt$$

$$= \eta\frac{\partial L}{\partial \dot{u}}\bigg|_I + \int_I \left(\frac{\partial L}{\partial u} + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{u}}\right)\right)\eta dt$$

Titik kritis u diperoleh jika :

$$\frac{d}{d\varepsilon}\mathcal{L}(u+\varepsilon\eta)\bigg|_{\varepsilon=0} = 0$$

$$\eta\frac{\partial L}{\partial \dot{u}}\bigg|_{I} = 0 \qquad \rightarrow \text{BCs}$$

$$\frac{\partial L}{\partial u} + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{u}}\right) = 0 \rightarrow \text{Persamaan Euler - Lagrange dari fungsional}$$

$$\mathcal{L}(u) = \int_I L(u,\dot{u},t)\,dt$$

Mekanika klasik

$$M = \{q | q : I \rightarrow R\},\ I \text{ interval waktu}$$

$$q = q(t) \quad : \text{Posisi}$$

$$\dot{q} = \frac{dq}{dt} \quad : \text{Kecepatan}$$

L adalah selisih antara energi kinetik dan energi potensial

$$\mathcal{L}(q) = \int_{I} L(q, \dot{q}, t)\,dt$$

$$\mathcal{L}(q) = \int_{I} (Ek - Ep)\,dt$$

$$\mathcal{L}(q) = \int_{I} \left(\tfrac{1}{2} m\dot{q}^2 - V(q,t)\right) dt$$

$$\mathcal{L}(q) = \int_I \left(\tfrac{1}{2} m\dot{q}^2 - V(q,t)\right) dt$$

Persamaan Euler Lagrange

$$\frac{\partial L}{\partial q} + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) = -\frac{\partial V}{\partial q} - m\ddot{q} = 0$$

$$\Leftrightarrow m\ddot{q} = -\frac{\partial V}{\partial q} = F$$

Jika momentum $p = m\dot{q}$

maka $\dfrac{dp}{dt} = m\ddot{q} = -\dfrac{\partial V}{\partial q}$

diperoleh

$$\dot{q} = p$$

$$\dot{p} = -\frac{\partial V}{\partial q}$$

Hamiltonian

$$H = Ek + Ep$$

$$H = 2Ek - L \quad (L = Ek - Ep)$$

$$L = 2Ek - H$$

Action Fungsional/Action Principle

$$\mathcal{L}(q, p) = \int_I L(q, p)dt$$

$$= \int_I \left(2Ek - H\right)dt$$

$$= \int_I \left(p\dot{q} - H(q, p)\right)$$

dengan $H(q, p) = \frac{1}{2} p^2 + V(q)$

Gunakan turunan berarah untuk mencari titik kritis dari $\mathcal{L}(q,p)$

$$\frac{d}{d\varepsilon}\mathcal{L}(q+\varepsilon\eta,p)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int p\frac{d}{dt}(q+\varepsilon\eta)\text{-}H(q+\varepsilon\eta,p)dt\Big|_{\varepsilon=0}$$

$$= p\eta\big|_I - \int_I\left(\frac{dp}{dt}+\frac{dH}{dq}\right)\eta dt$$

$$\frac{d}{d\varepsilon}\mathcal{L}(q,p+\varepsilon\eta)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int(p+\varepsilon\eta)\frac{dq}{dt}\text{-}H(q,p+\varepsilon\eta)dt\Big|_{\varepsilon=0}$$

$$= \int_I\left(\frac{dq}{dt}-\frac{dH}{dp}\right)\eta dt$$

Persamaan Euler Lagrange

$$\delta_q \mathcal{L} = 0 \quad dan \quad \delta_p \mathcal{L} = 0$$

$$sehingga \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

$$\partial_t \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix}$$

Energi Konservasi

$$H(q,\dot{q},t) = \tfrac{1}{2} m\dot{q}^2 + V(q,t), \text{ maka}$$

$$\frac{\partial H}{\partial t} = m\dot{q}\ddot{q} + \frac{\partial V}{\partial q}\dot{q} + \frac{\partial V}{\partial t} = \left(m\ddot{q} + \frac{\partial V}{\partial q} \right)\dot{q} + \frac{\partial V}{\partial t}$$

$$\text{Karena } m\ddot{q} + \frac{\partial V}{\partial q} = 0$$

$$\frac{\partial H}{\partial t} = \frac{\partial V}{\partial t}$$

$$\text{Jika } \frac{\partial V}{\partial t} = 0 \text{ maka } \frac{\partial H}{\partial t} = 0$$

Jika diketahui Hamiltonian : $H(q, p) = \int L(q, p)dx$

Action Principle: $\mathcal{L}(q, p) = \int_{I_2} \left(\int_{I_1} p\dot{q}dx - H(q, p) \right) dt$

maka sistem dinamik dari (q, p) :

$$\frac{dq}{dt} = \delta_p H$$

$$\frac{dp}{dt} = -\delta_q H$$

Contoh

Tentukan sistem dinamik dengan Hamiltonian :

$$H(u,\eta)=\int\left\{\frac{1}{2}g\eta^2+\frac{1}{2}(h+\eta)\left(u^2-\frac{1}{3}u_x^{\;2}\right)\right\}$$

Penyelesaian :

$$\frac{d}{d\varepsilon}H(u+\varepsilon\tau,\eta)\Big|_{\varepsilon=0}=\frac{d}{d\varepsilon}\int\left\{\frac{1}{2}g\eta^2+\frac{1}{2}(h+\eta)\left((u+\varepsilon\tau)^2-\frac{1}{3}\left(\frac{\partial}{\partial x}(u+\varepsilon\tau)\right)^2\right)\right\}dx\Bigg|_{\varepsilon=0}$$

$$=-\frac{1}{3}u_x(h+\eta)\tau\Big|_X+\int\left(u(h+\eta)+\frac{1}{3}(u_{xx}(h+\eta)+u_x(h_x+\eta_x))\right)\tau dx$$

$$\frac{d\eta}{dx}=\delta_u H(u,\eta)=u(h+\eta)+\frac{1}{3}(u_{xx}(h+\eta)+u_x(h_x+\eta_x))$$

$$\frac{d}{d\varepsilon}H(u,\eta+\varepsilon\tau)\bigg|_{\varepsilon=0}=\frac{d}{d\varepsilon}\int\left\{\frac{1}{2}g(\eta+\varepsilon\tau)^2+\frac{1}{2}(h+\eta+\varepsilon\tau)\left(u^2-\frac{1}{3}u_x{}^2\right)\right\}dx\bigg|_{\varepsilon=0}$$

$$=\int\left(g\eta+\frac{1}{2}u^2-\frac{1}{6}u_x{}^2\right)\tau dx$$

$$\frac{du}{dt}=\delta_\eta H(u,\eta)=g\eta+\frac{1}{2}u^2-\frac{1}{6}u_x{}^2$$

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