KOMPUTASI DAN DINAMIKA FLUIDA

TUGAS 2

Oleh

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1. Calculate the first variation of the following functionals:

(a) $$£(u)=\int_0^1 xu(x)^2+\left(\frac{\partial}{\partial x}u(x)\right)^2 dx$$

$$\frac{d}{d\varepsilon}£(u+\varepsilon\eta)\Big|_{\varepsilon=0}=\frac{d}{d\varepsilon}\int_0^1 x(u+\varepsilon\eta)^2+\left(\frac{\partial}{\partial x}(u+\varepsilon\eta)\right)^2 dx\Big|_{\varepsilon=0}$$

$$=\frac{d}{d\varepsilon}\int_0^1\left\{x\left(u^2+2\varepsilon u\eta+\varepsilon^2\eta^2\right)+\left(\partial_x u^2+2\varepsilon\partial_x u\partial_x\eta+\varepsilon^2(\partial_x\eta)^2\right)\right\}dx\Big|_{\varepsilon=0}$$

$$=\int_0^1\{2xu\eta+2\partial_x u\partial_x\eta\}dx$$

$$=\int_0^1 2xu\eta\ \ dx+\int_0^1 2\partial_x u\partial_x\eta\ \ dx$$

$$=\int_0^1 2xu\eta\ \ dx+2\eta\partial_x u\Big|_0^1-\int_0^1 2\eta\partial_{xx}u\ dx$$

$$=2\eta\partial_x u\Big|_0^1+\int_0^1(2xu-2\partial_{xx}u)\eta\ \ dx$$

Jadi, $\delta£(u)=2xu(x)-2\frac{\partial^2}{\partial x^2}u(x)$

(b) $$£(u)=\int_0^1\sin(x)u(x)^2+x^3\left(\frac{\partial}{\partial x}u(x)\right)^2 dx$$

$$\frac{d}{d\varepsilon}£(u+\varepsilon\eta)\Big|_{\varepsilon=0}=\frac{d}{d\varepsilon}\int_0^1\sin(x)(u+\varepsilon\eta)^2+x^3\left(\frac{\partial}{\partial x}(u+\varepsilon\eta)\right)^2 dx\Big|_{\varepsilon=0}$$

$$=\frac{d}{d\varepsilon}\int_0^1\sin(x)(u^2+2\varepsilon u\eta+\varepsilon^2\eta^2)+x^3\left(\partial_x u^2+2\varepsilon\partial_x u\partial_x\eta+\varepsilon^2(\partial_x\eta)^2\right)dx\Big|_{\varepsilon=0}$$

$$=\int_0^1 2u\eta\sin(x)+2x^3\partial_x u\partial_x\eta dx$$

$$=\int_0^1 2u\eta\sin(x)\ dx+\int_0^1 2x^3\partial_x u\partial_x\eta\ dx$$

$$=\int_0^1 2u\eta\sin(x)\ dx+2x^3\partial_x u\eta\Big|_0^1-\int_0^1\left(2x^3\partial_{xx}u+6x^2\partial_x u\right)\eta\ dx$$

$$=2x^3\partial_x u\eta\Big|_0^1+\int_0^1\left(2u\sin(x)-2x^3\partial_{xx}u-6x^2\partial_x u\right)\eta\ dx$$

Jadi, $\delta£(u)=2u(x)\sin(x)-2x^3\frac{\partial^2}{\partial x^2}u(x)-6x^2\frac{\partial}{\partial x}u(x)$

(c). $$£(u) = \int u(x)^2 + \left(\frac{\partial}{\partial x}u(x)\right)^2 dx$$

$$\frac{d}{d\varepsilon}£(u+\varepsilon\eta)\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int (u+\varepsilon\eta)^2 + \left(\frac{\partial}{\partial x}(u+\varepsilon\eta)\right)^2 dx\bigg|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\int (u^2 + 2\varepsilon u\eta + \varepsilon^2\eta^2) + \left(\partial_x u^2 + 2\varepsilon\partial_x u\partial_x\eta + \varepsilon^2(\partial_x\eta)^2\right)dx\bigg|_{\varepsilon=0}$$

$$= \int 2u\eta + 2\partial_x u\partial_x\eta\ dx$$

$$= \int 2u\eta\ dx + \int 2\partial_x u\partial_x\eta\ dx$$

$$= \int 2u\eta\ dx + 2\partial_x u\eta\big|_X - \int_0^1 2\partial_{xx}u\eta\ dx$$

$$= 2\partial_x u\eta\big|_x + \int_0^1 (2u - 2\partial_{xx}u)\eta\ dx$$

Jadi, $\delta£(u) = 2u(x) - 2\frac{\partial^2}{\partial x^2}u(x)$

(d) $$£(u) = \int_0^1 \left[\sin(u(x)) + \left(\frac{\partial^2}{\partial x^2}u(x)\right)^2\right]dx$$

$$\frac{d}{d\varepsilon}£(u+\varepsilon\eta)\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_0^1 \sin(u+\varepsilon\eta) + \left(\frac{\partial^2}{\partial x^2}(u+\varepsilon\eta)\right)^2 dx\bigg|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\int_0^1 (\sin(u)\cos(\varepsilon\eta) + \cos(u)\sin(\varepsilon\eta)) + \left((\partial_{xx}u)^2 + 2\varepsilon\partial_{xx}u\partial_{xx}\eta + \varepsilon^2(\partial_{xx}\eta)^2\right)dx\bigg|_{\varepsilon=0}$$

$$= \int_0^1 \eta\cos(u) + 2\partial_{xx}u\partial_{xx}\eta\ \ dx$$

$$= \int_0^1 \eta\cos(u)\ \ dx + 2\int_0^1 \partial_{xx}u\partial_{xx}\eta\ \ dx$$

$$= \int_0^1 \eta\cos(u)\ \ dx + 2\left(\partial_{xx}u\partial_x\eta\ \big|_0^1 - \int_0^1 \partial_{xxx}u\partial_x\eta\ \ dx\right)$$

$$= \int_0^1 \eta\cos(u)\ \ dx + 2\left(\partial_{xx}u\partial_x\eta\ \big|_0^1 - \left(\partial_{xxx}u\eta\ \big|_0^1 - \int_0^1 \eta\partial_{xxxx}u\ \ dx\right)\right)$$

$$= 2(\partial_{xx}u\partial_x\eta - \eta\partial_{xxx}u)\big|_0^1 + \int_0^1 (\cos(u) + 2\partial_{xxxx}u)\eta\ dx$$

Jadi, $\delta £(u) = \cos(u(x)) + 2\frac{\partial^4}{\partial x^4}u(x)$

(e) $$£(u) = \int_0^1 \left[u(x)^4 + \left(\frac{\partial}{\partial x} u(x) \right)^7 \right] dx$$

$$\frac{d}{d\varepsilon} £(u+\varepsilon\eta)\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_0^1 (u+\varepsilon\eta)^4 + \left(\frac{\partial}{\partial x}(u+\varepsilon\eta)\right)^7 dx\bigg|_{\varepsilon=0}$$

$$= \int_0^1 4u^3\eta + 7(\partial_x u)^6(\partial_x \eta)\ dx$$

$$= \int_0^1 4u^3\eta\ dx + \int_0^1 7(\partial_x u)^6(\partial_x \eta)\ dx$$

$$= \int_0^1 4u^3\eta\ dx + 7\left((\partial_x u)^6\eta\Big|_0^1 - \int_0^1 \eta\left(6(\partial_x u)^5 \partial_{xx}u\right)\ dx \right)$$

$$= 7(\partial_x u)^6\eta\Big|_0^1 + \int_0^1 \left(4u^3 - 42(\partial_x u)^5\partial_{xx}u\right)\eta\ \ dx$$

Jadi, $\delta £(u) = 4(u(x))^3 - 42\left(\frac{\partial}{\partial x}u(x)\right)^5 \frac{\partial^2}{\partial x^2}u(x)$

(f) $$£(u) = \int_0^1 n(x)\sqrt{1+\left(\frac{\partial}{\partial x}u(x)\right)^2}\ dx$$

$$\frac{d}{d\varepsilon} £(u+\varepsilon\eta)\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_0^1 n(x)\sqrt{1+\left(\frac{\partial}{\partial x}u(x+\varepsilon\eta)\right)^2}\ dx\bigg|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\int_0^1 n(x)\sqrt{1+(\partial_x u)^2 + 2\varepsilon\partial_x u\partial_x\eta + \varepsilon^2(\partial_x\eta)^2}\,dx\bigg|_{\varepsilon=0}$$

$$= \int_0^1 \frac{1}{2}n(x)(2\partial_x u\partial_x\eta)\left(1+(\partial_x u)^2 + 2\varepsilon\partial_x u\partial_x\eta + \varepsilon^2(\partial_x\eta)^2\right)^{-1/2}dx\bigg|_{\varepsilon=0}$$

$$= \int_0^1 \frac{n(x)}{\sqrt{1+(\partial_x u)^2}}(\partial_x u\partial_x\eta)dx$$

$$= \frac{n}{\sqrt{1+(\partial_x u)^2}}\partial_x u\eta\Bigg|_0^1 - \int_0^1\left(\frac{(\partial_x n\partial_x u + n\partial_{xx}u)\sqrt{1+(\partial_x u)^2} - n\partial_x u\left(\partial_x u\partial_{xx}u\left(1+(\partial_x u)^2\right)^{-1/2}\right)}{1+(\partial_x u)^2}\right)\eta dx$$

Jadi, $\delta £(u) = \dfrac{n(x)\left(\frac{\partial}{\partial x}u(x)\right)^2\left(\frac{\partial^2}{\partial x^2}u(x)\right)}{\left(1+\left(\frac{\partial}{\partial x}u(x)\right)^2\right)^{3/2}} - \dfrac{\left(\frac{\partial}{\partial x}n(x)\right)\left(\frac{\partial}{\partial x}u(x)\right)+n(x)\left(\frac{\partial^2}{\partial x^2}u(x)\right)}{\sqrt{1+\left(\frac{\partial}{\partial x}u(x)\right)^2}}$

(g) $$£(q) = \int_0^1 \left[\frac{1}{2}\dot{q}(t)^2 - \frac{1}{2}q(t)^2 + q(t)^3\right]dt$$

$$\frac{d}{d\varepsilon}£(q+\varepsilon\eta)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_0^1 \frac{1}{2}\partial_t(q+\varepsilon\eta)^2 - \frac{1}{2}(q+\varepsilon\eta)^2 + (q+\varepsilon\eta)^3\,dt\Big|_{\varepsilon=0}$$

$$= \int_0^1 \partial_t q\partial_t\eta - q\eta + 3q^2\eta\ \ dt$$

$$= \int_0^1 \partial_t q\partial_t\eta\ \ dt + \int_0^1 (3q^2 - q)\eta\ \ dt$$

$$= \partial_t q\eta\Big|_1^0 - \int_0^1 (\partial_{tt}q)\eta\ \ dt + \int_0^1 (3q^2 - q)\eta\ \ dt$$

$$= \partial_t q\eta\Big|_1^0 + \int_0^1 (3q^2 - q - \partial_{tt}q)\eta\ \ dt$$

Jadi, $\delta £(q) = 3(q(t))^2 - q(t) - \dfrac{\partial^2}{\partial t^2}q(t)$

(h) $$£(u) = \int_0^1 \left[\frac{1}{2}\partial_x u(x)^2 + x^3\sin(u(x)) + u(x)^5\right]dx$$

$$\frac{d}{d\varepsilon}£(u+\varepsilon\eta)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_0^1 \frac{1}{2}\left(\frac{\partial}{\partial x}(u+\varepsilon\eta)\right)^2 + x^3\sin(u+\varepsilon\eta) + (u+\varepsilon\eta)^5 dx\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\int_0^1 \frac{1}{2}(\partial_x u + \varepsilon\partial_x\eta)^2 + x^3(\sin(u)\cos(\varepsilon\eta) + \cos(u)\sin(\varepsilon\eta)) + (u+\varepsilon\eta)^5\,dx\Big|_{\varepsilon=0}$$

$$= \int_0^1 (\partial_x u\partial_x\eta + x^3\eta\cos(u) + 5u^4\eta)dx$$

$$= \int_0^1 (\partial_x u\partial_x\eta\ )dx + \int_0^1 (x^3\cos(u) + 5u^4)\eta\ dx$$

$$= \partial_x u\eta\Big|_0^1 - \int_0^1 (\partial_{xx}u\ )\eta\ dx + \int_0^1 (x^3\cos(u) + 5u^4)\eta\ dx$$

$$= \partial_x u\eta\Big|_0^1 + \int_0^1 (x^3\cos(u) + 5u^4 - \partial_{xx}u)\eta\ dx$$

Jadi, $\delta £(u) = x^3\cos(u(x)) + 5(u(x))^4 - \dfrac{\partial^2}{\partial x^2}u(x)$

2. Light rays, Fermat's principle

According to Fermat, the trajectory of a light ray between two points is such that the required time is as small as possible.

The propagation speed of light depends on material proprties, which is expressed by c_0/n where c_0 is the speed in vacuum (which is maximal), and n >1 is the so-called index of refraction, characteristic for the material.

For trajectories, simplicity described as graphs of functions $x \to y(x)$, the total time between points is

$$\int n(x,y)\sqrt{1+y_x{}^2}\,dx$$

This is also often called the *optical pathlength*. Note that this functional can also be given very different interpretations, depending on the meaning of n (for instance: the cost of a road between points when the local costs re given by n).

(a) Write down the Euler-Lagrange equations.

(b) Determine the optimal trajectory in case n does not depend on x eplicitly. Then use 'energy-conservation' to study trajectories.

(c) Consider the special case $n = y$ and $n = 1/y$ for which the trajectories can be expressed explicity.

<u>Penyelesaian:</u>

(a) Misalkan $\pounds(y) = \int n(x,y)\sqrt{1+y_x{}^2}\,dx$, maka:

$$\frac{d}{d\varepsilon}\pounds(y+\varepsilon\eta)\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int n(x,y+\varepsilon\eta)\sqrt{1+\left(\frac{\partial}{\partial x}(y+\varepsilon\eta)\right)^2}\,dx\Bigg|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\int n(x, y+\varepsilon\eta)\sqrt{1+\left(\frac{\partial}{\partial x}(y+\varepsilon\eta)\right)^2}\,dx\Bigg|_{\varepsilon=0}$$

$$= \int\left(\frac{\partial}{\partial y}n(x,y)\right)\eta\sqrt{1+y_x^{\,2}+2\varepsilon y_x\eta_x+\varepsilon^2\eta_x^{\,2}}+n(x,y+\varepsilon\eta)\frac{\left(y_x\eta_x+\varepsilon\eta_x^{\,2}\right)}{\sqrt{1+y_x^{\,2}+2\varepsilon y_x\eta_x+\varepsilon^2\eta_x^{\,2}}}\,dx\Bigg|_{\varepsilon=0}$$

$$= \int\left(\frac{\partial}{\partial y}n(x,y)\right)\eta\sqrt{1+y_x^{\,2}}+n(x,y)\frac{y_x\eta_x}{\sqrt{1+y_x^{\,2}}}\,dx$$

$$= \int\left(\left(\frac{\partial}{\partial y}n(x,y)\right)\sqrt{1+y_x^{\,2}}\right)\eta\;\;dx+\int n(x,y)\frac{y_x\eta_x}{\sqrt{1+y_x^{\,2}}}\,dx$$

$$= \int\left(\left(\frac{\partial}{\partial y}n(x,y)\right)\sqrt{1+y_x^{\,2}}\right)\eta\;\;dx+\eta\frac{n(x,y)y_x}{\sqrt{1+y_x^{\,2}}}\Bigg|_X-\int\left(\frac{\left(ny_{xx}+y_x(n_x+n_yy_x)\right)\sqrt{1+y_x^{\,2}}-ny_x\left(y_xy_{xx}\left(1+y_x^{\,2}\right)^{-1/2}\right)}{1+y_x^{\,2}}\right)\eta dx$$

$$= \eta\frac{n(x,y)y_x}{\sqrt{1+y_x^{\,2}}}\Bigg|_X+\int\left(\left(\frac{\partial}{\partial y}n(x,y)\right)\sqrt{1+y_x^{\,2}}+\frac{n(x,y)y_x^{\,2}y_{xx}}{\left(1+y_x^{\,2}\right)^{3/2}}-\frac{n(x,y)y_{xx}}{\sqrt{1+y_x^{\,2}}}-\frac{y_x(n_x+n_yy_x)}{\sqrt{1+y_x^{\,2}}}\right)\eta\;\;dx$$

Jadi persamaan Euler Lagrangenya adalah :

$$\left\{\left(\frac{\partial}{\partial y}n(x,y)\right)\sqrt{1+\left(\frac{\partial}{\partial x}y(x)\right)^2}+\frac{n(x,y)\left(\frac{\partial}{\partial x}y(x)\right)^2\left(\frac{\partial^2}{\partial x^2}y(x)\right)}{\left(1+\left(\frac{\partial}{\partial x}y(x)\right)^2\right)^{3/2}}-\frac{n(x,y)\left(\frac{\partial^2}{\partial x^2}y(x)\right)}{\sqrt{1+\left(\frac{\partial}{\partial x}y(x)\right)^2}}\right.$$

…(2.1)

$$\left.-\frac{\left(\frac{\partial}{\partial x}y(x)\right)\left(\frac{\partial}{\partial x}n(x,y)+\frac{\partial}{\partial y}n(x,y)\frac{\partial}{\partial x}y(x)\right)}{\sqrt{1+\left(\frac{\partial}{\partial x}y(x)\right)^2}}\right\}=0$$

(b) Secara umum, trajektori (light rays) diperoleh dengan mencari solusi dari persamaan :

$$\left\{ \left(\frac{\partial}{\partial y} n(x,y)\right)\sqrt{1+\left(\frac{\partial}{\partial x} y(x)\right)^2} + \frac{n(x,y)\left(\frac{\partial}{\partial x} y(x)\right)^2\left(\frac{\partial^2}{\partial x^2} y(x)\right)}{\left(1+\left(\frac{\partial}{\partial x} y(x)\right)^2\right)^{3/2}} - \frac{n(x,y)\left(\frac{\partial^2}{\partial x^2} y(x)\right)}{\sqrt{1+\left(\frac{\partial}{\partial x} y(x)\right)^2}} \right.$$

$$\left. - \frac{\left(\frac{\partial}{\partial x} y(x)\right)\left(\frac{\partial}{\partial x} n(x,y) + \frac{\partial}{\partial y} n(x,y)\frac{\partial}{\partial x} y(x)\right)}{\sqrt{1+\left(\frac{\partial}{\partial x} y(x)\right)^2}} \right\} = 0$$

Persamaan di atas dapat disederhanakan sebagai berikut :

Kedua ruas persamaan dikali dengan $\left(1+\left(\frac{\partial}{\partial x} y(x)\right)^2\right)^{3/2}$ sehingga diperoleh :

$$n_y\left(1+y_x^{\ 2}\right)^2 + ny_x^{\ 2}y_{xx} - ny_{xx}\left(1+y_x^{\ 2}\right) - y_x\left(n_x + n_y y_x\right)\left(1+y_x^{\ 2}\right) = 0$$

$$\Leftrightarrow n_y\left(1+2y_x^{\ 2}+y_x^{\ 4}\right) - ny_{xx} - n_x y_x\left(1+y_x^{\ 2}\right) - n_y y_x^{\ 2} - n_y y_x^{\ 4} = 0$$

$$\Leftrightarrow n_y\left(1+y_x^{\ 2}\right) - ny_{xx} - n_x y_x\left(1+y_x^{\ 2}\right) = 0$$

$$\Leftrightarrow n_y - n_x y_x - \frac{ny_{xx}}{\left(1+y_x^{\ 2}\right)} = 0$$

Dalam kasus, *n* tidak bergantung pada *x* secara eksplisit, berarti :

$$\frac{\partial}{\partial x} n(x,y) = 0$$

sehingga trajektori diperoleh dengan mencari solusi dari persamaan:

$$n_y - \frac{ny_{xx}}{\left(1+y_x^{\ 2}\right)} = 0 \qquad \ldots(2.2)$$

(c). **1. Kasus** $n = y$

Misalkan $\pounds(y) = \int y\sqrt{1 + y_x^{\ 2}}\,dx$,

maka dengan menggunakan persamaan (2.1) diperoleh :

$$\delta\pounds(y) = \sqrt{1+\left(\frac{\partial}{\partial x}y(x)\right)^2} + \frac{y(x)\left(\frac{\partial}{\partial x}y(x)\right)^2\left(\frac{\partial^2}{\partial x^2}y(x)\right)}{\left(1+\left(\frac{\partial}{\partial x}y(x)\right)^2\right)^{3/2}} - \frac{\left(\frac{\partial}{\partial x}y(x)\right)^2 + y(x)\left(\frac{\partial^2}{\partial x^2}y(x)\right)}{\sqrt{1+\left(\frac{\partial}{\partial x}y(x)\right)^2}},$$

dan dengan menggunakan persamaan (2.2) diperoleh :

$$1 - \frac{y(x)y_{xx}}{\left(1+y_x^{\ 2}\right)} = 0 \text{ atau } \left(1+y_x^{\ 2}\right) - y(x)y_{xx} = 0 \qquad \ldots(2.3)$$

Trajektori dari ‘light ray’ untuk kasus $n = y$ ditentukan oleh solusi dari persamaan (2.3), yaitu:

$$y_1(x) = \frac{1+\left(e^{(C1x)}\right)^2\left(\left(e^{(C1)(C2)}\right)^2\right)}{2C1e^{(C1x)}e^{(C1)(C2)}}, \text{ dan } y_2(x) = \left(\frac{1+\left(e^{(C1x)}\right)^2\left(e^{(C1)(C2)}\right)^2}{2C1}\right)e^{(xC1)}e^{(C1)(C2)}$$

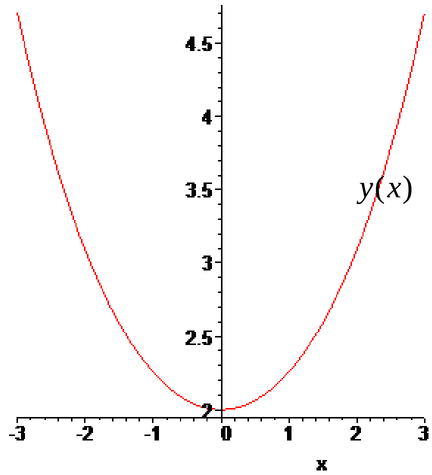
C1 dan *C2* adalah konstanta yang dapat diperoleh jika syarat awal diketahui.

Contoh:

Misalkan diketahui syarat awal $y(0) = 2$ dan $y'(0) = 0$, maka diperoleh :

$$\mathrm{y}(x) = \left(1 + \frac{1}{\left(\mathbf{e}^{\left(\frac{x}{2}\right)}\right)^2}\right)\mathbf{e}^{\left(\frac{x}{2}\right)}, \ \mathrm{y}(x) = \left(1 + \frac{1}{\left(\mathbf{e}^{\left(-\frac{x}{2}\right)}\right)^2}\right)\mathbf{e}^{\left(-\frac{x}{2}\right)}$$

$$\mathrm{y}(x) = \frac{1 + \left(\mathbf{e}^{\left(\frac{x}{2}\right)}\right)^2}{\mathbf{e}^{\left(\frac{x}{2}\right)}}, \ \mathrm{y}(x) = \frac{1 + \left(\mathbf{e}^{\left(-\frac{x}{2}\right)}\right)^2}{\mathbf{e}^{\left(-\frac{x}{2}\right)}}$$



Gambar trajektori

2. Kasus $n = 1/y$

Misalkan $\pounds(y) = \int 1/y \sqrt{1 + y_x^{\,2}}\, dx$,

maka dengan menggunakan persamaan (2.1) diperoleh :

$$\delta\pounds(y) = -\frac{\sqrt{1+\left(\frac{\partial}{\partial x}y(x)\right)^2}}{y(x)^2} + \frac{\left(\frac{\partial}{\partial x}y(x)\right)^2\left(\frac{\partial^2}{\partial x^2}y(x)\right)}{y(x)\left(1+\left(\frac{\partial}{\partial x}y(x)\right)^2\right)^{3/2}} - \frac{\left(\frac{\partial^2}{\partial x^2}y(x)\right)}{y(x)\sqrt{1+\left(\frac{\partial}{\partial x}y(x)\right)^2}} + \frac{\left(\frac{\partial}{\partial x}y(x)\right)^2}{y(x)^2\sqrt{1+\left(\frac{\partial}{\partial x}y(x)\right)^2}} ,$$

dan dengan menggunakan persamaan (2.2) diperoleh :

$$y\left(1 + y_x^{\,2}\right) + y^2 y_{xx} = 0 \qquad \text{…(2.4)}$$

Trajektori dari ‘light ray’ untuk kasus $n = 1/y$ ditentukan oleh solusi dari persamaan (2.4), yaitu:

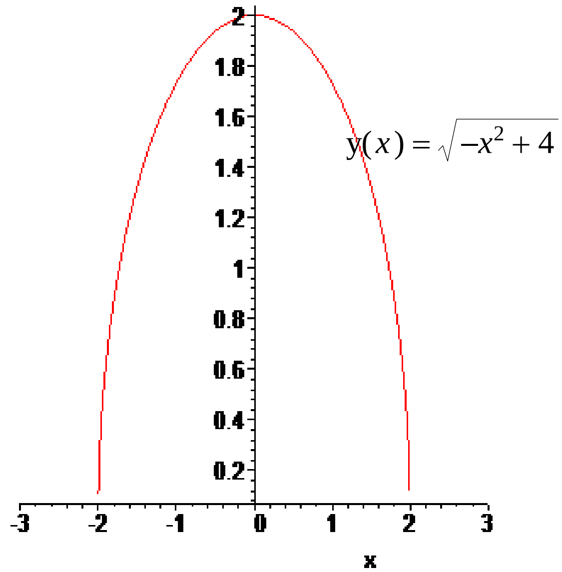
$$y(x) = \pm\sqrt{-x^2 - 2ax + 2b} ,$$

a,*b* adalah konstanta yang dapat diperoleh jika syarat awal diberikan.

Contoh:

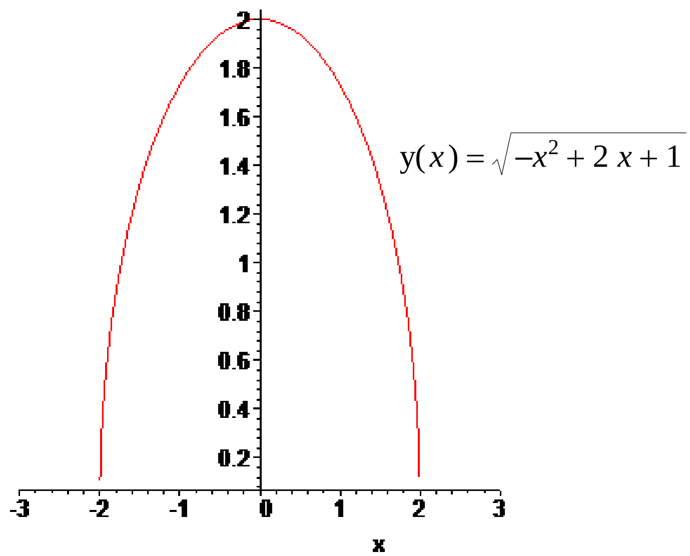
1. Misalkan diketahui syarat awal $y(0) = 2$ dan $y'(0) = 0$,maka diperoleh :

$$y(x) = \sqrt{-x^2 + 4}$$



2. Misalkan diketahui syarat awal $y(0) = 1$ dan $y'(0) = 1$,maka diperoleh :

$$y(x) = \sqrt{-x^2 + 2\,x + 1}$$



3. Boussinesq type of equations

Surface waves (in one horizontal direction *x*) that decay at infinity $\left(|x| \mapsto \pm\infty\right)$ can be described in terms of the wave height $\eta(x,t)$ and a velocity $u(x,t)$ in the following form (a Hamiltonian system):

$$\partial_t u = \partial_x \{\delta_\eta H(u,\eta)\},$$
$$\partial_t \eta = -\partial_x \{\delta_u H(u,\eta)\}$$

for a suitable functional (the Hamiltonian) $H(u,\eta)$.

(a) Describe the equations in full detail when the Hamiltonian is given by the following functional

$$H(u,\eta) = \int \left\{ \frac{1}{2} g\eta^2 + \frac{1}{2}(h+\eta)\left(u^2 - \frac{1}{3}u_x^{\ 2}\right)\right\} dx$$

(This set of equations are the 'linearized' equations.)

(b) In another case (shallow water, no dispersion, but nonlinear), the equations are of the form

$$\partial_t u = \partial_x \left\{ g\eta + \frac{1}{2}u^2 \right\},$$
$$\partial_t \eta = -\partial_x \{u + \beta\eta u\},$$

where β is a constant. Determine the value of β such that this system of equation is a Hamiltonian system of the form (3,4) given above.

(c) Show that the equations have the horizontal momentum as constant of the motion:

$$\int u(x)\eta(x)dx$$

Penyelesaian:

(a) $H(u,\eta)=\int\left\{\frac{1}{2}g\eta^2+\frac{1}{2}(h+\eta)\left(u^2-\frac{1}{3}u_x^{\ 2}\right)\right\}dx$

- $\langle\delta_u H(u,\eta),\tau\rangle=\frac{d}{d\varepsilon}H(u+\varepsilon\tau,\eta)\Big|_{\varepsilon=0}=\frac{d}{d\varepsilon}\int\left\{\frac{1}{2}g\eta^2+\frac{1}{2}(h+\eta)\left((u+\varepsilon\tau)^2-\frac{1}{3}\left(\frac{\partial}{\partial x}(u+\varepsilon\tau)\right)^2\right)\right\}dx\Big|_{e=0}$

$$=\frac{1}{2}\int\frac{d}{d\varepsilon}\left\{g\eta^2+(h+\eta)\left(\left(u^2+2\varepsilon u\tau+\varepsilon^2\tau^2\right)-\frac{1}{3}\left(u_x+2\varepsilon u_x\tau_x+\varepsilon^2\tau_x^{\ 2}\right)\right)\right\}dx\Big|_{e=0}$$

$$=\frac{1}{2}\int(h+\eta)\left(2u\tau-\frac{1}{3}(2u_x\tau_x)\right)dx$$

$$=\int u(h+\eta)\tau dx-\int\frac{1}{3}u_x(h+\eta)\tau_x dx$$

$$=\int u(h+\eta)\tau dx-\frac{1}{3}\left\{u_x(h+\eta)\tau\big|_X-\int\left(u_{xx}(h+\eta)+u_x(h_x+\eta_x)\right)\tau\, dx\right\}$$

$$=-\frac{1}{3}u_x(h+\eta)\tau\Big|_X+\int\left(u(h+\eta)+\frac{1}{3}\left(u_{xx}(h+\eta)+u_x(h_x+\eta_x)\right)\right)\tau dx$$

Jadi, $\delta_u H(u,\eta)=u(h+\eta)+\frac{1}{3}\left(u_{xx}(h+\eta)+u_x(h_x+\eta_x)\right)$

- $\langle\delta_\eta H(u,\eta),\tau\rangle=\frac{d}{d\varepsilon}H(u,\eta+\varepsilon\tau)\Big|_{\varepsilon=0}=\frac{d}{d\varepsilon}\int\left\{\frac{1}{2}g(\eta+\varepsilon\tau)^2+\frac{1}{2}\left(h+(\eta+\varepsilon\tau)\right)\left(u^2-\frac{1}{3}u_x^{\ 2}\right)\right\}dx\Big|_{e=0}$

$$=\frac{d}{d\varepsilon}\int\left\{\frac{1}{2}g\left(\eta^2+2\varepsilon\eta\tau+\varepsilon^2\tau^2\right)+\frac{1}{2}\left(h+(\eta+\varepsilon\tau)\right)\left(u^2-\frac{1}{3}u_x^{\ 2}\right)\right\}dx\Big|_{e=0}$$

$$=\int\left\{g(\eta\tau)+\frac{1}{2}\tau\left(u^2-\frac{1}{3}u_x^{\ 2}\right)\right\}dx$$

$$=\int\left(g\eta+\frac{1}{2}u^2-\frac{1}{6}u_x^{\ 2}\right)\tau dx$$

Jadi, $\delta_\eta H(u,\eta)=g\eta+\frac{1}{2}u^2-\frac{1}{6}u_x^{\ 2}$

Selanjutnya :

$$\partial_t u=-\partial_x\left\{\delta_\eta H(u,\eta)\right\}=-\partial_x\left\{g\eta+\frac{1}{2}u^2-\frac{1}{6}u_x^{\ 2}\right\}$$

$$=-\left\{g\eta_x+uu_x-\frac{1}{3}u_xu_{xx}\right\}$$

$$\partial_t \eta = \partial_x \{\delta_u H(u,\eta)\} = \partial_x \left\{ u(h+\eta) + \frac{1}{3}\left(u_{xx}(h+\eta) + u_x(h_x+\eta_x)\right) \right\}$$

$$= u_x(h+\eta) + u(h_x+\eta_x) + \frac{1}{3}\left(u_{xxx}(h+\eta) + u_{xx}(h_x+\eta_x) + u_{xx}(h_x+\eta_x) + u_x(h_{xx}+\eta_{xx})\right)$$

$$= u_x(h+\eta) + u(h_x+\eta_x) + \frac{1}{3}\left(u_{xxx}(h+\eta) + 2u_{xx}(h_x+\eta_x) + u_x(h_{xx}+\eta_{xx})\right)$$

(b). Diketahui persamaan :

$$\partial_t u = \partial_x \left\{ g\eta + \frac{1}{2}u^2 \right\}, \qquad \ldots(3.1)$$
$$\partial_t \eta = -\partial_x \{u + \beta\eta u\},$$

harus memenuhi memenuhi persamaan

$$\partial_t u = \partial_x \{\delta_\eta H(u,\eta)\}, \qquad \ldots(3.2)$$
$$\partial_t \eta = -\partial_x \{\delta_u H(u,\eta)\}$$

Berarti :

$$\delta_\eta H(u,\eta) = g\eta + \frac{1}{2}u^2 \text{ dan } \delta_u H(u,\eta) = u + \beta\eta u$$

Dengan menggunakan rumus langsung untuk memperoleh Euler Lagrange, diperoleh:

$$\frac{\partial H}{\partial \eta} - \frac{d}{dx}\left(\frac{\partial H}{\partial \dot{\eta}}\right) = \int g\eta + \frac{1}{2}u^2 dx$$

Karena $\delta_\eta H(u,\eta)$ tidak memuat $\dot{\eta}$, berarti $\frac{d}{dx}\left(\frac{\partial H}{\partial \dot{\eta}}\right) = 0$, sehingga :

$$\frac{\partial H}{\partial \eta} = \int g\eta + \frac{1}{2}u^2 dx$$

Akibatnya :

$$H(u,\eta) = \int\left(\int g\eta + \frac{1}{2}u^2 dx\right)d\eta$$

$$= \iint g\eta + \frac{1}{2}u^2 d\eta dx$$

$$= \int\left(\frac{1}{2}g\eta^2 + \frac{1}{2}u^2\eta + c\right)dx$$

$$\delta_u H(u,\eta) = \frac{\partial}{\partial u}\left(\frac{1}{2}g\eta^2 + \frac{1}{2}u^2\eta + c\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \dot{u}}\left(\frac{1}{2}g\eta^2 + \frac{1}{2}u^2\eta + c\right)\right)$$

$$= (u\eta + \partial_u c) - 0$$

$$= u\eta + \partial_u c$$

Karena $\delta_u H(u,\eta) = u + \beta\eta u = u\eta + \partial_u c$, maka haruslah $\beta = 1$ dan $\partial_u c = u$.

$\partial_u c = u$, maka $c = \frac{1}{2}u^2$.

Jadi, $H(u,\eta) = \int \left\{ \frac{1}{2} g\eta^2 + \frac{1}{2}(\eta + 1)u^2 \right\} dx$.

(c). Diketahui :

$$\partial_t u = \partial_x \left\{ g\eta + \frac{1}{2}u^2 \right\} = g\eta_x + uu_x$$

$$\partial_t \eta = -\partial_x \{u(1+\eta)\} = -(u_x(1+\eta) + u\eta_x)$$

Maka:

$$\frac{d}{dt}\int u(x)\eta(x)dx = \int (u_t\eta + \eta_t u)dx$$

$$= \int (g\eta_x + uu_x)\eta - (u_x(1+\eta) + u\eta_x)u\, dx$$

$$= \int g\eta_x\eta - u^2\eta_x - uu_x\, dx$$

$$= \int (g\eta - u^2)\eta_x\, dx - \int uu_x\, dx$$

$$= (g\eta - u^2)\eta + C_1 - \int (g\eta_x - 2uu_x)\eta dx - \int uu_x\, dx$$

$$= (g\eta - u^2)\eta + C_1 - \int g\eta\eta_x\, dx + \int (2u\eta - u)u_x\, dx$$

$$\int g\eta\eta_x\, dx = g\eta^2 + C_2 - \int g\eta_x\eta\, dx$$

$$\Leftrightarrow \int g\eta\eta_x\, dx = \frac{1}{2}g\eta^2 + \frac{C_2}{2}$$

$$\int (2u\eta - u)u_x\, dx = (2u\eta - u)u + C_3 - \left(\int 2u_x\eta + 2u\eta_x - u_x\right)u\, dx$$

➢ $$= (2u^2\eta - u^2) + C_3 - \int (2u\eta - u)u_x dx - 2\int u\eta_x\, dx$$

$$\Leftrightarrow \int (2u\eta - u)u_x\, dx = u^2\eta - \frac{u^2}{2} + C_3 - \int u\eta_x\, dx$$

Sehingga:

$$\frac{d}{dt}\int u(x)\eta(x)dx = (g\eta - u^2)\eta + C_1 - \left(\frac{1}{2}g\eta^2 + \frac{C_2}{2}\right) + \left(u^2\eta - \frac{u^2}{2} + C_3 - \int u\eta_x\, dx\right)$$

$$= \frac{1}{2}g\eta^2 - \frac{u^2}{2} - \int u\eta_x\, dx + C$$