KOMPUTASI DAN DINAMIKA FLUIDA

TUGAS 3

Oleh

RIRIN SISPIYATI
NIM : 20106003
Program Studi Matematika

INSTITUT TEKNOLOGI BANDUNG
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The orthogonality constraints in the successive characterization are natural constraints: although essential in the definition of the constraint set, there is no effect in the equation for the critical point: the corresponding multiplier vanishes. To verify this, consider the equation for

$$\psi \in Crit\{\mathcal{R}(u) | u \in H, \mathcal{N}(u, f) = 0\}$$

where f is any given function. The governing equation is for some multipliers μ,σ

$$L\psi = \mu N\psi + \sigma f, \text{ with } \mu = \mathcal{R}(\psi).$$

Verify that σ = 0 if f is some eigenfunction, but that in general σ will not vanish.

Penyelesaian:

$$\psi \in Crit\{\mathcal{R}(u) | u \in H, \mathcal{N}(u, f) = 0\}$$

Juga bisa kita tulis

$\psi \in Crit\{\mathcal{Q}(u) | u \in H, \mathcal{N}(u) = 1, \mathcal{N}(u, f) = 0\}$ dengan f adalah sembarang fungsi.

Nilai eigen yang bersesuaian:

$$\mu = R(\psi) = \frac{Q(u)}{\mathcal{N}(\psi)}$$

Dari *Multiply-Lagrange Rule* maka ada μ,σ sedemikian sehingga

$$L\psi = \mu N\psi + \sigma f$$

$$\langle L\psi, f\rangle = \mu\langle \mathrm{N}\psi, f\rangle + \sigma\langle f, f\rangle$$

$$\Leftrightarrow Q\langle \psi, f\rangle = \mu N\langle \psi, f\rangle + \sigma\langle f, f\rangle$$

f dan ψ adalah fungsi eigen dari nilai eigen yang berbeda, maka dari proposisi 3.8 maka ψ dan f ortogonal terhadap bentuk kuadratik Q dan N, yaitu

$$Q\langle \psi, f\rangle = 0 \Leftrightarrow N\langle \psi, f\rangle = 0$$

sehingga

$$\sigma\langle f, f\rangle = 0$$

Karena f fungsi eigen maka $f \neq 0$ sehingga $\sigma = 0$

Secara umum jika f bukan fungsi eigen dan $N(u, f) = 0$ maka

$$Q\langle \psi, f\rangle = \sigma\langle f, f\rangle$$

Jadi

$Q\langle \psi, f\rangle \neq 0$ dan $\sigma \neq 0$

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The special case $\rho \equiv 1, p \equiv 1, q \equiv 0$ *provides Fourier theory (for function that are odd on* $[-\pi,\pi]$*): then the eigenvalues and (normalized) corresponding eigenfunction are given by*

$$\lambda_m = m^2, \varphi_m = \sqrt{2/\pi}\sin mx, \quad m \ge 1.$$

The completeness result in the spectrual theorem implies that any function satisfying the boundary conditions can be written as a Fourier-sine series

$$u(x) = \sqrt{2/\pi}\sum_{1}^{\infty} u_m \sin mx,$$

For Fourier coefficients given by

$$u_m = \langle u, \varphi_m \rangle = \sqrt{2/\pi}\int u(x \sin mxdx);$$

The convergence in the 𝒩-norm is just the usual L_2*-norm:*

$$\int \left(u - \sum_{1}^{M} u_m \varphi_m(x) \right)^2 dx \to 0, \textit{ for } M \to \infty.$$

The convergence in the Q-norm implies a much stronger statement. To investigate that, exploit the Poincare inequality: for some constant $c_1 > 0$ *it holds that*

$$|u|_\infty^2 \le c_1 \int u_x^2 \quad \text{for all } u, u(0) = u(\pi) = 0$$

Then the convergence in the Q-norm implies the pointwise convergence of the Fourier-sine series:

$$\left| u - \sum_{1}^{M} u_m \varphi_m(x) \right|_\infty \to 0, \textit{ for } M \to \infty$$

Penyelesaian :

Sturm-Liouville eigenvalue problem:

$$L\varphi = -\partial_x (p(x)\varphi_x) + q(x)\varphi = \lambda \rho(x)\varphi, \qquad \varphi(0) = \varphi(1) = 0$$

$$\mathcal{N}(\text{u}) = \int \rho(x) u^2 dx, \qquad \mathcal{Q}(\text{u}) = \int \left[p(x)u_x^2 + q(x)u^2 \right] dx$$

$$U_0 = \{ u \in L_2 | u(0) = u(\pi) = 0 \}$$

Untuk kasus khusus dimana $\rho \equiv 1, p \equiv 1, q \equiv 0$ maka

$$L\varphi = -\partial_x(\varphi_x) - \lambda\varphi \text{, dengan } \varphi(0) = \varphi(\pi) = 0$$

Diperoleh

$$\partial_{xx}\varphi + \lambda\varphi = 0 \qquad (*)$$

Dalam mencari solusi persamaan (*), ada tiga kemungkinan: Nilai eigennya mungkin negative, nol, atau positif.

Kasus 1 (Nilai eigen negatif)

Misal $\lambda = -v^2$

diperoleh

$$\partial_{xx}\varphi - v^2\varphi = 0$$

Persamaan Karakteristik:

$$r^2 = v^2 \rightarrow r_{1,2} = \pm v$$

Solusi umumnya:

$$\varphi(x) = c_1 e^{vx} + c_2 e^{-vx}$$

$$\varphi(0) = c_1 + c_2 = 0 \Leftrightarrow c_1 = -c_2$$

$$\varphi(\pi) = c_1 e^{v\pi} + c_2 e^{-v\pi} = c_1 e^{v\pi} - c_1 e^{-v\pi} = c_1(e^{v\pi} - e^{-v\pi}) = 2c_1 \sinh v\pi$$

Untuk $v > 0, c_1 = 0, c_2 = 0$, maka solusi yang didapatkan solusi yang trivial: $\varphi(x) = 0$

Kasus 2 (Nilai eigen nol)

$$\lambda = 0$$

diperoleh

$$\partial_{xx}\varphi(x) = 0$$

$$\Leftrightarrow \partial_x\varphi(x) = c_1$$

$$\Leftrightarrow \varphi(x) = c_1 x + c_2$$

Gunakan Boundary Condition

$$\varphi(0) = c_1(0) + c_2 = 0 \Leftrightarrow c_2 = 0$$

$$\varphi(\pi) = c_1\pi = 0 \Leftrightarrow c_1 = 0$$

Jadi solusinya juga trivial: $\varphi(x) = 0$

Kasus 3 (Nilai eigen positif)

Misal $\lambda = v^2$

diperoleh

$$\partial_{xx}\varphi + v^2\varphi = 0$$

Persamaan Karakteristik:

$$r^2 = -v^2 \rightarrow r_{1,2} = \pm iv$$

Solusi umumnya:

$$\varphi(x) = c_1 \cos vx + c_2 \sin vx$$

$$\varphi(0) = c_1 = 0 \Leftrightarrow c_1 = 0$$

$$\varphi(\pi) = c_2 \sin v\pi = 0$$

Untuk solusi nontrivial $(c_2 \neq 0)$ maka

$$\sin v\pi = 0$$

Karena kita tahu bahwa $\sin m\pi = 0, \forall m = 0,1,2,3,...$

maka $v = m$ untuk $m = 1,2,3,...$ (m=0 memberikan solusi trivial)

Kita tulis

$$L\varphi_m = -\lambda_m \varphi_m$$

$\varphi_m(x) = c_2 \sin mx$, $m = 1,2,3,...$

$\varphi_m(x)$ adalah fungsi eigen

Normalisasi dari $\varphi_m(x)$ berarti $\|\varphi_m(x)\| = 1$

$$1 = \|\varphi_m(x)\| = \langle \varphi_m, \varphi_m \rangle = \int_0^\pi (c_2 \sin mx)(c_2 \sin mx)dx$$

$$= c_2^2 \int_0^\pi \sin^2 mx$$

$$= c_2^2 \left[\tfrac{x}{2} - \tfrac{\sin 2mx}{4m}\right]_0^\pi$$

$$= c_2^2 \left(\tfrac{\pi}{2}\right)$$

$$1 = c_2^2 \left(\tfrac{\pi}{2}\right)$$

$$c_2^2 = \frac{2}{\pi}$$

$$c_2 = \sqrt{\frac{2}{\pi}}$$

Sehingga $\varphi_m(x) = \sqrt{\tfrac{2}{\pi}} \sin mx, m \geq 1$ adalah basis ortonormal yang bersesuaian dengan $\lambda_m = m^2$

Misalkan $u \in U_0$ yang diekspansi dengan menggunakan basis $\varphi_m(x) = \sqrt{\tfrac{2}{\pi}} \sin mx$

Maka $u(x)$ dapat ditulis :

$$u(x) = \sum_1^\infty u_m \varphi_m(x) = \sqrt{\tfrac{2}{\pi}} \sum_1^\infty u_m \sin mx$$

Dimana

$$u_m = \frac{\langle u, \varphi_m \rangle}{\langle \varphi_m, \varphi_m \rangle} = \langle u, \varphi_m \rangle = \sqrt{\tfrac{2}{\pi}} \int_0^\pi u(x) \sin mx dx \qquad (\langle \varphi_m, \varphi_m \rangle = 1)$$

Kekonvergenan dalam norm-$\mathcal{N}$ sama halnya dengan norm-L_2:

$$\mathcal{N}\left(u - \sqrt{\tfrac{2}{\pi}} \sum_1^M u_m \sin mx \right) \to 0 \Leftrightarrow \int \left(u - \sqrt{\tfrac{2}{\pi}} \sum_1^M u_m \sin mx \right)^2 \to 0 \text{ untuk } M \to \infty$$

Poincare inequality: untuk $c_1 > 0$ maka berlaku

$$|u|_\infty^2 \le c_1 \int u_x^2 \quad \text{untuk semua } u, u(0) = u(\pi) = 0$$

maka

$$Q\left(u - \sqrt{\tfrac{2}{\pi}}\sum_{1}^{M} u_m \sin mx \right) \to 0 \Leftrightarrow \int \partial_x^2 \left(u - \sqrt{\tfrac{2}{\pi}}\sum_{1}^{M} u_m \sin mx \right) \to 0 \text{ untuk } M \to \infty$$

karena $\int \partial_x^2 \left(u - \sqrt{\tfrac{2}{\pi}}\sum_{1}^{M} u_m \sin mx \right) \to 0$ maka berdasarkan Poincare inequality

$$\left| u - \sqrt{\tfrac{2}{\pi}}\sum_{1}^{M} u_m \sin mx \right|_{\infty} \to 0 \text{ untuk } M \to \infty .$$

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For given smooth and bounded functions p(x) and q(x), consider the quadratic forms

$$Q(u) = \int_0^1 p(x)u_x^2 + q(x)u^2, \quad \mathcal{N}(u) = \int_0^1 u^2$$

(a) Write down the Sturm-Liouille eigenvalue corresponding to Q and N on the set U_0 of function from $C^1([0,1])$ with u(1) = 0.

(b) Show that if the function p in (1) is strictly positive on the entire interval [0,1], the Rayleigh quotient

$$\mathcal{R}(u) := \frac{Q(u)}{\mathcal{N}(u)}$$

is bounded from below on U_0.

Penyelesaian:

a. Eigen Value Problem(EVP):

$$Lu = \lambda Nu$$

$$Q(u) = \langle Lu, u \rangle$$

$$\mathcal{N}(u) = \langle Lu, u \rangle$$

Kita peroleh Lu dari $\delta Q(u) = 2Lu$

$$\delta Q(u;v) = \frac{d}{d\varepsilon}\int_0^1 p(x)\partial_x(u+\varepsilon v)^2 + q(x)(u+\varepsilon v)^2\,dx \bigg|_{\varepsilon=0}$$

$$= \int_0^1 (2p(x)\partial_x u \partial_x v + 2q(x)uv)dx$$

$$= 2p(x)v\partial_x u\Big|_0^1 + \int_0^1 (-2\partial_x(p(x)\partial_x u) + 2q(x)u)vdx$$

$$= 0 + \int_0^1 (-2\partial_x(p(x)\partial_x u) + 2q(x)u)vdx$$

maka diperoleh

$$\delta Q(u) = -2\partial_x(p(x)\partial_x u) + 2q(x)u = 2Lu$$

sehingga

$$Lu = -\partial_x(p(x)\partial_x u) + q(x)u$$

Kita peroleh N dari $\delta\mathcal{N}(u) = 2Nu$

$$\delta\mathcal{N}(u;v) = \frac{d}{d\varepsilon}\int_0^1 (u+\varepsilon v)^2 \bigg|_{\varepsilon=0}$$

$$= 2\int_0^1 uvdx$$

Maka kita peroleh

$$\delta\mathcal{N}(u) = 2u = 2Nu$$

maka

N = 1

Sehingga

EVP: $-\partial_x(p(x)\partial_x u) + q(x)u = \lambda u$

b. Jika $p(x)$ fungsi bernilai positif pada interval [0,1], Rayleigh quotient

$$\mathcal{R}(u) := \frac{Q(u)}{\mathcal{N}(u)}$$

Dalam dimensi tak hingga Rayleigh quotient memenuhi terbatas di bawah atau terbatas di atas. Jika $q(x)$ diasumsikan fungsi yang terbatas dan $p(x)$ fungsi bernilai positif pada interval [0,1], maka $Q(u)$ definit positif sehingga $\mathcal{R}(u)$ terbatas di bawah.

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A functional of Sturm-Liouville type:

$$\mathcal{L}(u) = \int \left[p(x)(\partial_x u)^2 + q(x)u^2 - 2f(x)u \right] dx$$

Akan dicari solusi hampiran u dengan metode Ritz-Galerkin.

Penyelesaian:

$$\mathcal{L}(u) = \int_0^L \left[p(x)(\partial_x u)^2 + q(x)u^2 - 2f(x)u \right] dx$$

Persamaan Euler-Lagrange:

$$\delta\mathcal{L}(u) = -\partial_x(p(x)\partial_x u) + q(x)u - f(x) = 0$$

$$\varepsilon(u) = 0 \Leftrightarrow \delta L(u) = 0$$

$$u(x) \approx \tilde{u}(x) = \sum_{j=1}^{N} a_j \phi_j(x)$$

$$\varepsilon(\tilde{u}) = 0$$

$$\left\langle \varepsilon\left(\sum_{k=1}^{N} a_k \phi_k(x)\right), \phi_m(x) \right\rangle = 0 \qquad , \mathrm{m} = 1, \ldots \mathrm{N}$$

$$\Leftrightarrow \left\langle -\partial_x\left(p(x)\partial_x\left(\sum_{k=1}^{N} a_k \phi_k(x)\right)\right) + q(x)\left(\sum_{k=1}^{N} a_k \phi_k(x)\right) - f(x), \phi_m(x) \right\rangle = 0$$

$$\Leftrightarrow \int_I -\partial_x\left(p(x)\partial_x\left(\sum_{k=1}^{N} a_k \phi_k(x)\right)\right)\phi_m(x) + \int_I q(x)\left(\sum_{j=1}^{N} a_k \phi_m(x)\phi_k(x)\right)dx - \int_I f(x)\phi_m(x)dx = 0$$

$$\Leftrightarrow -\sum_{j=1}^{N} a_k\, p(x)\phi_m(x)\partial_x(\phi_k(x))\Big|_I + \int_I p(x)\left(\sum_{j=1}^{N} a_k (\partial_x \phi_m(x))(\partial_x \phi_k(x))\right)dx + \int_I q(x)\left(\sum_{j=1}^{N} a_k (\phi_m(x))(\phi_k(x))\right)dx$$
$$- \int_I f(x)\phi_m(x)dx = 0$$

$$\Leftrightarrow \int_I p(x)\left(\sum_{j=1}^{N} a_k (\partial_x \phi_m(x))(\partial_x \phi_k(x))\right)dx + \int_I q(x)\left(\sum_{j=1}^{N} a_k (\phi_m(x))(\phi_k(x))\right)dx - \int_I f(x)\phi_m(x)dx = 0$$

Jika

$$P_{mk} = \int p(x)(\partial_x \phi_m)(\partial_x \phi_k)dx$$

$$Q_{mk} = \int q(x)(\phi_m)(\phi_k)dx$$

$$F_m = \int f(x)(\phi_m)dx$$

Maka

$\left[P_{mk}\right]\overline{a}+\left[Q_{mk}\right]\overline{a}-\left[F_m\right]=0$ (N buah persamaan)

Sehingga diperoleh $a_1, a_2, ..., a_N$

didapat $\tilde{u}(x)=\sum_{k=1}^{N}\overline{a}_k\phi_k$

5. S-L-Eigenvalue problem

Diketahui masalah nilai eigen:

$-\partial_x p(x)\partial_x u+q(x)u=\lambda\rho(x)u$

Akan dicari solusi hampiran u dengan metode Ritz-Galerkin.

Penyelesaian:

$$\varepsilon(u)=0 \Leftrightarrow \delta L(u)=0$$

$$u(x)\approx\tilde{u}(x)=\sum_{j=1}^{N}a_j\phi_j(x)$$

$$\varepsilon(\tilde{u})=0$$

$$\left\langle \varepsilon\left(\sum_{k=1}^{N}a_k\phi_k(x)\right),\phi_m(x)\right\rangle=0 \qquad ,\mathrm{m}=1,...\mathrm{N}$$

$$\Leftrightarrow\left\langle -\partial_x\left(p(x)\partial_x\left(\sum_{k=1}^{N}a_k\phi_k(x)\right)\right)+q(x)\left(\sum_{k=1}^{N}a_k\phi_k(x)\right)-\lambda\rho(x)\left(\sum_{k=1}^{N}a_k\phi_k(x)\right),\phi_m(x)\right\rangle=0$$

$$\Leftrightarrow\int_I-\partial_x\left(p(x)\partial_x\left(\sum_{k=1}^{N}a_k\phi_k(x)\right)\right)\phi_m(x)+\int_I q(x)\left(\sum_{j=1}^{N}a_k\phi_m(x)\phi_k(x)\right)dx-\int_I\lambda\rho(x)\sum_{k=1}^{N}a_k\phi_m(x)\phi_k(x)dx=0$$

$$\Leftrightarrow-\sum_{k=1}^{N}a_k\,p(x)\phi_m(x)\partial_x(\phi_k(x))\Big|_I+\int_I p(x)\left(\sum_{k=1}^{N}a_k(\partial_x\phi_m(x))(\partial_x\phi_k(x))\right)dx+\int_I q(x)\left(\sum_{k=1}^{N}a_k(\phi_m(x))(\phi_k(x))\right)dx$$

$$-\int_I\lambda\rho(x)\sum_{k=1}^{N}a_k\phi_m(x)\phi_k(x)dx=0$$

$$\Leftrightarrow-\int_I p(x)\left(\sum_{j=1}^{N}a_k(\partial_x\phi_m(x))(\partial_x\phi_k(x))\right)dx+\int_I q(x)\left(\sum_{j=1}^{N}a_k(\phi_m(x))(\phi_k(x))\right)dx-\int_I\lambda\rho(x)\sum_{k=1}^{N}a_k\phi_m(x)\phi_k(x)dx=0$$

Jika

$$P_{mk}=\int_0^L p(x)(\partial_x\phi_m)(\partial_x\phi_k)dx$$

$$Q_{mk}=\int_0^L q(x)(\phi_m)(\phi_k)dx$$

$$R_{mk}=\int_0^L \rho(x)(\phi_m)(\phi_k)dx$$

Maka

$[P_{mk}]\overline{a} + [Q_{mk}]\overline{a} - \lambda[R_{mk}]\overline{a} = 0$

Sehingga diperoleh $a_1, a_2, ..., a_N$

didapat $\tilde{u}(x) = \sum_{k=1}^{N} \overline{a}_k \phi_k$