SISTEM DINAMIK

TUGAS 1

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3.1 Consider the two-dimensional system in R^2

$$\dot{x} = y\left(1 + x - y^2\right)$$
$$\dot{y} = x\left(1 + y - x^2\right)$$

Determine the critical points and characterise the linearised flow in a neighbourhood of these points.

Penyelesaian:

$$\dot{x} = y\left(1 + x - y^2\right) = y + xy - y^3$$
$$\dot{y} = x\left(1 + y - x^2\right) = x + xy - x^3$$

Titik kritis:

1. Untuk persamaan $1 + x - y^2 = 0$,

   jika $x = 0$ maka $\quad y\left(1 - y^2\right) = 0$

   $\Leftrightarrow\ y = 0$ atau $y = \pm 1$

   Sehingga titik kritisnya adalah $\left(0,0\right)$ dan $\left(0,\pm 1\right)$

2. Untuk persamaan $1 + y - x^2 = 0$,

   jika $y = 0$ maka $\quad \mathrm{x}\left(1 - x^2\right) = 0$

   $\Leftrightarrow\ x = 0$ atau $x = \pm 1$

   Sehingga titik kritisnya adalah $\left(0,0\right)$ dan $\left(\pm 1,0\right)$

3. Untuk persamaan $1 + x - y^2 = 0$ dan $1 + y - x^2 = 0$

   $1 + x - y^2 = 0 \rightarrow x = y^2 - 1 \ldots \left(1\right)$ $y = +, x = +$

   $\Leftrightarrow 1 + y - x^2 = 0$

   $\Leftrightarrow \left(1 + y\right) - \left(y^2 - 1\right)^2 = 0$

   $\Leftrightarrow \left(1 + y\right) - \left(y^2 - 1\right)\left(y^2 - 1\right) = 0$

   $\Leftrightarrow \left(1 + y\right) - \left(y + 1\right)^2\left(y - 1\right)^2 = 0$

   $\Leftrightarrow \left(1 + y\right)\left[1 - \left(y + 1\right)\left(y - 1\right)^2\right] = 0$

   $\Leftrightarrow \left(1 + y\right)\left[1 - y^3 + y^2 + y - 1\right] = 0$

   $\Leftrightarrow \left(1 + y\right)\left(-y^3 + y^2 + y\right) = 0$

   $\Leftrightarrow \left(1 + y\right)\left(-y\right)\left(y^2 - y - 1\right) = 0$

$$\Leftrightarrow y_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\Leftrightarrow x_1 = \left(\frac{1+\sqrt{5}}{2} \right)^2 - 1 = \left(\frac{1+2\sqrt{5}+5}{4} \right) - 1 = \frac{1+\sqrt{5}}{2}$$

$$\Leftrightarrow x_2 = \left(\frac{1-\sqrt{5}}{2} \right)^2 - 1 = \left(\frac{1-2\sqrt{5}+5}{4} \right) - 1 = \frac{1-\sqrt{5}}{2}$$

Matriks Jacobian:

$$J = \begin{pmatrix} y & 1+x-3y^2 \\ 1+y-3x^2 & x \end{pmatrix}$$

1) Untuk titik (0,0) $\rightarrow$ $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

   Persamaan Karakteristik = $\lambda^2 - 1 = 0 \rightarrow \lambda_1 = 1$ atau $\lambda_2 = -1$

   Diperoleh nilai eigen 1 dan -1 sehingga (0,0) adalah saddle point

2) Untuk titik (0,1) $\rightarrow$ $J = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$

   Persamaan karakteristik : $-\lambda + \lambda^2 + 4 = 0 \Leftrightarrow \lambda^2 - \lambda + 4 = 0$

   Sehingga didapat $\lambda_{1,2} = \frac{1 \pm \sqrt{1-4\cdot 4}}{2} = \frac{1 \pm \sqrt{-15}}{2}$

   Jadi, titik (0,1) adalah titik focus dengan negative atraktor.

3) Untuk titik (0,-1) $\rightarrow$ $J = \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix}$

   Persamaan karakteristik : $\lambda + \lambda^2 = 0 \Leftrightarrow \lambda(1+\lambda) = 0$ maka $\lambda_1 = 0$ atau $\lambda_2 = -1$

   Karena salah satu nilai eigennya mengandung 0, maka titik (0,-1) degenerate.

4) Untuk titik (-1,0) $\rightarrow$ $J = \begin{pmatrix} 0 & 0 \\ -2 & -1 \end{pmatrix}$

   Persamaan karakteristik : $\lambda^2 + \lambda = 0 \Leftrightarrow \lambda(\lambda + 1) = 0$ maka $\lambda_1 = 0$ atau $\lambda_2 = -1$

   Karena salah satu nilai eigennya mengandung 0, maka titik (-1,0) degenerate.

5) Untuk titik (1,0) $\rightarrow$ $J = \begin{pmatrix} 0 & 2 \\ -2 & 1 \end{pmatrix}$, $u = x - 1$ dan $v = y$

   Persamaan karakteristik : $\lambda^2 - \lambda + 4 = 0$

Sehingga didapat $\lambda_{1,2} = \frac{1 \pm \sqrt{1-4\cdot 4}}{2} = \frac{1 \pm \sqrt{-7}}{2}$

Titik (0,1) adalah titik focus dengan negative atraktor.

6) Untuk titik $\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\ ,\ \frac{1}{2}+\frac{1}{2}\sqrt{5}\right) \to J = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}\sqrt{5} & -1,431 \\ -1,431 & 1,618 \end{pmatrix}$

Persamaan Karakteristik : $(1,618-\lambda)^2 - 1,431^2 = 0$

$$\Leftrightarrow \lambda^2 - 3,236\lambda + 2,618 - 2,048 = 0$$

$$\Leftrightarrow \lambda^2 - 3,236\lambda + 0,57 = 0$$

$$\lambda_{1,2} = \frac{3,236 \pm \sqrt{10,47-2,28}}{2}$$

$$= \frac{3,236}{2} \pm \frac{1}{2}\cdot(2,862)$$

$\lambda_{1,2} \to$ Bernilai real dan berbeda tanda

Sehingga titik $\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\ ,\ \frac{1}{2}+\frac{1}{2}\sqrt{5}\right)$ adalah saddle point.

7) Untuk titik $\left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\ ,\ \frac{1}{2}-\frac{1}{2}\sqrt{5}\right) \to J = \begin{pmatrix} \frac{1}{2}-\frac{1}{2}\sqrt{5} & -2,42 \\ -2,42 & -0,914 \end{pmatrix}$

Persamaan Karakteristik: $(-0,914-\lambda)^2 - 2,42^2 = 0$

$$\Leftrightarrow \lambda^2 + 1,828\lambda + 0,835 - 5,856 = 0$$

$$\Leftrightarrow \lambda^2 + 1,828\lambda - 5,021 = 0$$

$$\lambda_{1,2} = \frac{-1,828 \pm \sqrt{3,342+20,084}}{2}$$

$$= \frac{-1,828}{2} \pm \frac{1}{2}\cdot(4,84)$$

$\lambda_{1,2} \to$ Bernilai real dan berbeda tanda

Sehingga titik $\left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\ ,\ \frac{1}{2}-\frac{1}{2}\sqrt{5}\right)$ adalah saddle point.

3.2 We are studying the geometrical properties of the critical point (0,0) of the autonomous system

$$\dot{x} = ax + by + f(x,y)$$

$$\dot{y} = cx + dy + g(x,y)$$

with $f\left(x,y\right)=0\left(r\right)$,$g\left(x,y\right)$=0 as $r=\sqrt{x^2+y^2}\to 0$.

a. Suppose that (0,0) is a focus in the linearised system. Show that in the full system, the orbits are also spiralling near (0,0) i.e. for each orbit near (0,0) the polar angle $\theta\left(t\right)$ takes all values in $\left[0,2\pi\right)$ for $t>a$ and each $a\in R$.

b. Can we prove a similar statement for node-like behaviour. Analyse the case $a=-1, b=0, c=0, d=-1,\ f\left(x,y\right)=-y/\log\left(x^2+y^2\right)^{1/2}$,
$g\left(x,y\right)=x/\log\left(x^2+y^2\right)^{1/2}$

Penyelesaian:

a. Karena (0,0) adalah focus maka nilai eigen-nya:

$\lambda_{1,2}=s\pm i\cdot t$, dan bentuk linearisasinya

$$\begin{pmatrix}\dot{x}\\ \hline \dot{y}\end{pmatrix}=\begin{pmatrix}s & t\\ -t & s\end{pmatrix}\begin{pmatrix}x\\ \hline y\end{pmatrix}\quad s,t>0$$

<u>Cek</u> : $\left(s-\lambda\right)^2+t=0$

$$\Leftrightarrow\ s^2-2s\lambda+\lambda^2+t^2=0$$

$$\Leftrightarrow\ \lambda^2-2s\lambda+s^2+t^2=0$$

$$\lambda_{1,2}=\frac{2s\pm\sqrt{4s^2-4\left(1\right)\left(t^2+5\right)}}{2}=s\pm i\cdot t$$

berarti $a=d$, $-c=b$

$$\dot{x}=ax+by$$
$$\dot{y}=-bx+ay$$

misal $r^2=x^2+y^2$, $\tan\theta={}^{y}\!/\!_{x}$

$$rr'=x\dot{x}+y\dot{y}$$
$$\Leftrightarrow rr'=x\left(ax+by\right)+y\left(-bx+ay\right)$$
$$\Leftrightarrow rr'=ax^2+ay^2$$
$$\Leftrightarrow r'=ar$$
$$\therefore r\left(t\right)=Ce^{at}$$

$$r^2\theta' = x\dot{y} - y\dot{x}$$
$$\Leftrightarrow r^2\theta' = x(-bx + ay) - y(ax + by)$$
$$\Leftrightarrow r^2\theta' = -bx^2 - by^2$$
$$\Leftrightarrow r^2\theta' = -br^2$$
$$\Leftrightarrow \theta' = -b$$
$$\therefore \theta(t) = -bt + C$$

Jika $a > 0$ dan b>0, maka $r(t) \to \infty$ untuk t $\to \infty$. $\theta(t)$ bergerak berlawanan arah jarum jam sehingga orbit yang diperoleh adalah focus dengan atraktor negative.

b. $f(x, y) = -y/\log\left(x^2 + y^2\right)^{1/2}$ dan $g(x, y) = x/\log\left(x^2 + y^2\right)^{1/2}$, maka

$$\dot{x} = -x + -y/\log\left(x^2 + y^2\right)^{1/2}$$
$$\dot{y} = -y + x/\log\left(x^2 + y^2\right)^{1/2}$$

Misal $x = r\cos\theta$ dan $y = r\sin\theta$

$$r'\cos\theta - r\theta'\sin\theta = -r\cos\theta - r\sin\theta/\log r$$
$$r'\sin\theta - r\theta'\cos\theta = -r\sin\theta - r\cos\theta/\log r$$

Diperoleh

$$r' = -r \Leftrightarrow \frac{dr}{dt} = -r$$
$$\Leftrightarrow \frac{1}{r}dr = -dt$$
$$\Leftrightarrow \ln r = -t + C$$
$$\therefore r(t) = Ce^{-t}$$

$$\sec^2\theta\theta' = \frac{\left(-r\sin\theta + {r\cos\theta}/{\log r}\right)r\cos\theta + r\sin\theta\left(r\cos\theta + {r\sin\theta}/{\log r}\right)}{r^2\cos^2\theta}$$
$$= \frac{{r^2}/{\log r}}{r^2\cos^2\theta}$$
$$= \frac{1}{\log r}\sec^2\theta$$

$$\theta' = \frac{1}{\log r}$$
$$\theta = K + \log(-\log C) - \log(t - \log C)$$

3.3 Consider the system

$$\dot{x} = 16x^2 + 9y^2 - 25$$
$$\dot{y} = 16x^2 - 16y^2$$

a. Determine the critical point and characterise them by linearisation.
b. Sketch the phase-flow.

Penyelesaian:

Titik Kritis:

$16(x^2 + y^2) = 0 \Leftrightarrow x^2 = y^2$

$$16x^2 + 9y^2 = 25$$
$$\Leftrightarrow 16x^2 + 9x^2 = 25$$
$$\Leftrightarrow 25x^2 = 25$$
$$\Leftrightarrow x^2 = 1$$

Maka didapat $x = \pm 1$ dan $y = \pm 1$

Sehingga titik kritisnya $(1,1), (1,\text{-}1), (-1,1),$ dan $(-1,-1)$

Matriks Jacobian :

$$J = \begin{pmatrix} 32x & 18y \\ 32x & -32y \end{pmatrix}$$

1. Untuk titik $(1,1)$ bentuk linearisasinya

$$d\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 32 & 18 \\ 32 & -32 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \text{ dengan } u = x-1 \text{ dan } v = y-1$$

$$J = \begin{pmatrix} 32 & 18 \\ 32 & -32 \end{pmatrix}$$

Persamaan karakteristik : $\lambda^2 - 1600 = 0 \Leftrightarrow \lambda^2 = 1600$ maka $\lambda_{1,2} = \pm 40$

Karena kedua nilai eigen real tetapi berbeda tanda maka titik $(1,1)$ adalah saddle point tanpa atraktor

2. Untuk titik $(1,-1)$ bentuk linearisasinya

$$d\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 32 & -18 \\ 32 & 32 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \text{ dengan } u = \text{ dan } v =$$

$$J = \begin{pmatrix} 32 & -18 \\ 32 & 32 \end{pmatrix}$$

Persamaan karakteristik : $\lambda^2 - 64\lambda + 1600 = 0$

Sehingga didapat $\lambda_{1,2} = \dfrac{64 \pm \sqrt{4096 - 6400}}{2} = \dfrac{64 \pm \sqrt{2304}}{2}$

Jadi, titik (1,-1) adalah titik focus dengan negative atraktor.

3. Untuk titik $(-1,1)$ bentuk linearisasinya

$d\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -32 & 18 \\ -32 & -32 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ dengan $u =$ dan $v =$

$J = \begin{pmatrix} -32 & 18 \\ -32 & -32 \end{pmatrix}$

Persamaan karakteristik : $\lambda^2 + 64\lambda + 1600 = 0$

Sehingga didapat $\lambda_{1,2} = \dfrac{-64 \pm \sqrt{4096 - 6400}}{2} = \dfrac{-64 \pm \sqrt{2304}}{2}$

Jadi, titik $(-1,1)$ adalah titik focus dengan positive atraktor.

4. Untuk titik $(-1,-1)$ bentuk linearisasinya

$d\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -32 & -18 \\ -32 & 32 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ dengan $u =$ dan $v =$

$J = \begin{pmatrix} -32 & -18 \\ -32 & 32 \end{pmatrix}$

Persamaan karakteristik : $\lambda^2 - 1600 = 0 \Leftrightarrow \lambda^2 = 1600$ maka $\lambda_{1,2} = \pm 40$

Karena kedua nilai eigen real tetapi berbeda tanda maka titik $(-1,-1)$ adalah saddle point tanpa atraktor.

3.4 Consider the system

$$\dot{x} = -y + x(x^2 + y^2)\sin\sqrt{x^2 + y^2}$$
$$\dot{y} = x + y(x^2 + y^2)\sin\sqrt{x^2 + y^2}$$

a Determine the number of critical points. Characterise (0,0) by linear analysis.

b. Is (0,0) an attractor?

Penyelesaian:

$$\dot{x} = -y + x(x^2 + y^2)\sin\sqrt{x^2 + y^2}$$
$$\dot{y} = x + y(x^2 + y^2)\sin\sqrt{x^2 + y^2}$$

$$f(x,y) = -y + x(x^2+y^2)\sin\sqrt{x^2+y^2} = 0$$
$$\Leftrightarrow (x^2+y^2)\sin\sqrt{x^2+y^2} = {}^{y}\!/\!_{x}$$
$$g(x,y) = x + y(x^2+y^2)\sin\sqrt{x^2+y^2} = 0$$
$$\Leftrightarrow x + y\left(\frac{y}{x}\right) = 0$$
$$\Leftrightarrow x^2 + y^2 = 0$$

Didapat titik kritisnya adalah (0,0).

$$f(x,y) = -y + x(x^2+y^2)\sin\sqrt{x^2+y^2}$$
$$f_x(x,y) = (3x^2+y^2)\sin\sqrt{x^2+y^2} + (x^3+xy^2)\cos\sqrt{x^2+y^2}\ \tfrac{1}{2}(x^2+y^2)^{\frac{1}{2}}.2x$$
$$f_y(x,y) = -1 + (2y)\sin\sqrt{x^2+y^2} + (x^3+xy^2)\cos\sqrt{x^2+y^2}\ \tfrac{1}{2}(x^2+y^2)^{\frac{1}{2}}.2x$$
$$g(x,y) = x + y(x^2+y^2)\sin\sqrt{x^2+y^2}$$
$$g_x(x,y) = 1 + (2xy)\sin\sqrt{x^2+y^2} + (x^3+xy^2)\cos\sqrt{x^2+y^2}\ \tfrac{1}{2}(x^2+y^2)^{\frac{1}{2}}.2x$$
$$g_y(x,y) = (x^2+3y^2)\sin\sqrt{x^2+y^2} + (x^3+xy^2)\cos\sqrt{x^2+y^2}\ \tfrac{1}{2}(x^2+y^2)^{\frac{1}{2}}.2x$$

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

Jika (0,0) disubstitusi pada matriks Jacobian maka

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Sehingga persamaan karakteristiknya : $\lambda^2 + 1 = 0 \Leftrightarrow \lambda_{1,2} = \pm\sqrt{-1}$

Jadi titik (0,0) adalah center.

Persamaan (*) diubah dalam bentuk polar dengan $x = r\cos\theta$ dan $y = r\sin\theta$, maka

$$\left.\begin{array}{l}(1)\ r'\cos\theta - r\theta'\sin\theta = -r\sin\theta + r^3\cos\theta\sin r \\ (2)\ r'\sin\theta + r\theta'\cos\theta = r\cos\theta + r^3\sin\theta\sin r\end{array}\right. \ldots(**)$$

Jika persamaan (1) dikali dengan $(-\sin\theta)$ dan persamaan (2) dikali dengan $(\cos\theta)$, maka

$$(1)\ \text{-}r'\sin\theta\cos\theta - r\theta'\sin^2\theta = -r\sin^2\theta + r^3\cos\theta\sin\theta\sin r$$
$$(2)\ r'\sin\theta\cos\theta + r\theta'\cos^2\theta = r\cos^2\theta + r^3\cos\theta\sin\theta\sin r$$

Sehingga didapat $r\theta' = r \Leftrightarrow \theta' = 1$

Dengan mensubstitusi $\theta' = 1$ pada persamaan (**) maka diperoleh

$$r' = r^3\cos^2\theta\sin r + r^3\sin^2\theta\sin r \Leftrightarrow r' = r^3\sin r$$