TEORI KONTROL OPTIMUM

TUGAS

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2.2-5 Comparison of Different Discrete Controllers

$$x_{k+1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} x_k + \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} u_k , \quad x_0 = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

a. Find the open-loop control u_0, u_1 to drive the initial state to $x_2 = 0$ while minimizing the cost

$$J_a = \frac{1}{2} \sum_{k=0}^{1} u_k^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} u_k$$

Check your answer by "simulation" (i.e., apply your u_0, u_1 to the plant to verify that $x_2 = 0$).

Penyelesaian :

- Persamaan state :

$$x_{k+1} = Ax_k + Bu_k ,$$

dimana : $A = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix}; \quad B = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$

- Hamiltonian:

$$H = \frac{1}{2} \sum_{k=0}^{1} u_k^T R u_k + p_{k+1}^T \{Ax_k + Bu_k\},$$

dimana: $R = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

- Persamaan Costate:

$$p_k = \frac{\partial H}{\partial x_k} = A^T p_{k+1} = \begin{bmatrix} -\frac{1}{2} & 3 \\ \frac{1}{2} & 1 \end{bmatrix} p_{k+1}$$

- Stationary:

$$0 = \frac{\partial H}{\partial u_k} = Ru_k + B^T p_{k+1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} u_k + \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} p_{k+1}$$

sehingga:

$$u_k = -R^{-1} B^T p_{k+1} = -\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} p_{k+1} = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} p_{k+1} = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \left(A^T\right)^{N-k-1} p_N$$

$N = 2$, maka :

$$u_k = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \left(A^T\right)^{1-k} p_2$$

- $x_k = A^k x_0 - \sum_{i=0}^{k-1} A^{k-i-1} B R^{-1} B^T \left(A^T\right)^{1-i} p_2$

Sehingga :

$$x_2 = A^2 x_0 - \sum_{i=0}^{1} A^{1-i} BR^{-1} B^T \left(A^T\right)^{1-i} p_2$$

$$G_{0,2} = \sum_{i=0}^{1} A^{1-i} BR^{-1} B^T \left(A^T\right)^{1-i} = \left(ABR^{-1} B^T A^T\right) + \left(BR^{-1} B^T\right)$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 3 \\ \frac{1}{2} & -1 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{3}{2} \\ -\frac{3}{2} & 17 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{13}{4} & \frac{1}{2} \\ \frac{1}{2} & 19 \end{bmatrix}$$

Maka :

$$p_2 = \left(G_{0,2}\right)^{-1} \left(A^2 x_0 - x_2\right) = \begin{bmatrix} \frac{38}{123} & \frac{-1}{123} \\ \frac{-1}{123} & \frac{13}{246} \end{bmatrix} \left\{ \begin{bmatrix} \frac{7}{4} & \frac{-3}{4} \\ \frac{-9}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} 8 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} \frac{38}{123} & \frac{-1}{123} \\ \frac{-1}{123} & \frac{13}{246} \end{bmatrix} \begin{bmatrix} 11 \\ -26 \end{bmatrix} = \begin{bmatrix} \frac{148}{41} \\ \frac{-60}{41} \end{bmatrix}$$

- $u_0 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \left(A^T\right) p_2 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 3 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} \frac{148}{41} \\ \frac{-60}{41} \end{bmatrix} = \begin{bmatrix} \frac{120}{41} \\ \frac{254}{41} \end{bmatrix}$
- $u_1 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} p_2 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{148}{41} \\ \frac{-60}{41} \end{bmatrix} = \begin{bmatrix} \frac{-88}{41} \\ \frac{-148}{41} \end{bmatrix}$

Cek /simulasi :

- $x_1 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} x_0 + \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} u_0 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{120}{41} \\ \frac{254}{41} \end{bmatrix} = \begin{bmatrix} -2 \\ 20 \end{bmatrix} + \begin{bmatrix} \frac{494}{41} \\ \frac{240}{41} \end{bmatrix} = \begin{bmatrix} \frac{412}{41} \\ \frac{1060}{41} \end{bmatrix}$
- $x_2 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} u_1 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \frac{412}{41} \\ \frac{1060}{41} \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{-88}{41} \\ \frac{-148}{41} \end{bmatrix} = \begin{bmatrix} \frac{324}{41} \\ \frac{176}{41} \end{bmatrix} + \begin{bmatrix} \frac{-324}{41} \\ \frac{-176}{41} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

b. Find a constant state variable feedback to input component one of the form

$$u_k^1 = -Kx_k$$

where $u_k = \begin{bmatrix} u_k^1 & u_k^2 \end{bmatrix}^T$, to yield a deadbeat control (all closed-loop poles at the origin). Find the closed-loop state trajectory.

Penyelesaian :

Misalkan :

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

Maka :

$$u_0^1 = -Kx_0 = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \end{bmatrix} \Rightarrow \frac{120}{41} = -8k_1 - 4k_2 \qquad ...(1)$$

$$u_1^1 = -Kx_1 = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} \frac{412}{41} \\ \frac{1060}{41} \end{bmatrix} \Rightarrow 88 = -412k_1 - 1060k_2 \quad ...(2)$$

maka dari (1) dan (2) diperoleh $k_1 = -0,40259$ dan $k_2 = 0,0734$.

Jadi, $K = \begin{bmatrix} -0,40259 & 0,0734 \end{bmatrix}$

c. Let $J_c = 10x_2^T x_2 + J_a$, with J_a as in part a. Solve the Riccatti equation to determine the optimal control u_0, u_1. Find the optimal cost.

Penyelesaian:

$$J_c = 10x_2^T x_2 + \frac{1}{2}\sum_{k=0}^{1} u_k^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} u_k ,$$

berarti : $S_2 = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}$; $Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $R = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Persamaan Riccatti:

$$S_k = Q + A^T S_{k+1}\left(I + BR^{-1}B^T S_{k+1}\right)^{-1} A = A^T S_{k+1}\left(I + BR^{-1}B^T S_{k+1}\right)^{-1} A$$

Maka :

$$S_1 = A^T S_2 \left(I + BR^{-1}B^T S_2\right)^{-1} A$$

$$= \begin{bmatrix} -\frac{1}{2} & 3 \\ \frac{1}{2} & -1 \end{bmatrix}\begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}\right)^{-1}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 60 \\ 10 & -20 \end{bmatrix}\begin{bmatrix} 61 & 40 \\ 40 & 41 \end{bmatrix}^{-1}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 60 \\ 10 & -20 \end{bmatrix}\begin{bmatrix} \frac{41}{901} & \frac{-40}{901} \\ \frac{-40}{901} & \frac{-461}{901} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{13585}{901} & \frac{-5465}{901} \\ \frac{-5465}{901} & \frac{2225}{901} \end{bmatrix}$$

Kontrol Optimal :

$$u_k = -R^{-1}B^T S_{k+1}\left(I + BR^{-1}B^T S_{k+1}\right)^{-1} Ax_k$$

Maka:

- $u_0 = -R^{-1}B^T S_1\left(I + BR^{-1}B^T S_1\right)^{-1} Ax_0$

$$= \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \frac{13585}{901} & \frac{-5465}{901} \\ \frac{-5465}{901} & \frac{2225}{901} \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \frac{13585}{901} & \frac{-5465}{901} \\ \frac{-5465}{901} & \frac{2225}{901} \end{bmatrix}\right)^{-1}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-8120}{901} & \frac{3240}{901} \\ \frac{-13585}{901} & \frac{5465}{901} \end{bmatrix}\begin{bmatrix} \frac{30726}{901} & \frac{-11945}{901} \\ \frac{16240}{901} & \frac{-5579}{901} \end{bmatrix}^{-1}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-8120}{901} & \frac{3240}{901} \\ \frac{-13585}{901} & \frac{5465}{901} \end{bmatrix}\begin{bmatrix} \frac{-797}{3578} & \frac{11945}{25046} \\ \frac{-1160}{1789} & \frac{15363}{12523} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{36520}{12523} \\ \frac{11005}{1789} \end{bmatrix} \approx \begin{bmatrix} 2{,}916 \\ 6{,}151 \end{bmatrix}$$

- $u_1 = -R^{-1}B^T S_2 \left(I + BR^{-1}B^T S_2\right)^{-1} Ax_0$

$$= \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}\right)^{-1}\begin{bmatrix} -\tfrac{1}{2} & \tfrac{1}{2} \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -20 & -20 \\ -20 & 0 \end{bmatrix}\begin{bmatrix} \tfrac{41}{901} & \tfrac{-40}{901} \\ \tfrac{-40}{901} & \tfrac{-461}{901} \end{bmatrix}\begin{bmatrix} -\tfrac{1}{2} & \tfrac{1}{2} \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \tfrac{-20}{901} & \tfrac{-420}{901} \\ \tfrac{-820}{901} & \tfrac{800}{901} \end{bmatrix}\begin{bmatrix} -\tfrac{1}{2} & \tfrac{1}{2} \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \tfrac{-80400}{36941} \\ \tfrac{-124880}{36941} \end{bmatrix} \approx \begin{bmatrix} -2{,}176 \\ -3{,}380 \end{bmatrix}$$

- $x_1 = Ax_0 + Bu_0$

$$= \begin{bmatrix} -\tfrac{1}{2} & \tfrac{1}{2} \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} \tfrac{36520}{12523} \\ \tfrac{11005}{1789} \end{bmatrix} = \begin{bmatrix} -2 \\ 20 \end{bmatrix} + \begin{bmatrix} \tfrac{150075}{12523} \\ \tfrac{73040}{12523} \end{bmatrix} = \begin{bmatrix} \tfrac{125029}{12523} \\ \tfrac{323500}{12523} \end{bmatrix} \approx \begin{bmatrix} 9{,}984 \\ 25{,}832 \end{bmatrix}$$

- $x_2 = Ax_1 + Bu_1$

$$= \begin{bmatrix} -\tfrac{1}{2} & \tfrac{1}{2} \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 9{,}984 \\ 25{,}832 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} -2{,}176 \\ -3{,}380 \end{bmatrix} = \begin{bmatrix} 7{,}924 \\ 4{,}119 \end{bmatrix} + \begin{bmatrix} -7{,}733 \\ -4{,}353 \end{bmatrix} = \begin{bmatrix} 0{,}191 \\ -0{,}234 \end{bmatrix}$$

Optimal Cost

$$J_c = 10x_2^T x_2 + \frac{1}{2}\sum_{k=0}^{1} u_k^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} u_k$$

$$= 10\begin{bmatrix} 0{,}191 & -0{,}234 \end{bmatrix}\begin{bmatrix} 0{,}191 \\ -0{,}234 \end{bmatrix} + \frac{1}{2}\left\{\begin{bmatrix} 2{,}916 & 6{,}151 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2{,}916 \\ 6{,}151 \end{bmatrix} + \begin{bmatrix} -2{,}176 & -3{,}380 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2{,}176 \\ -3{,}380 \end{bmatrix}\right\}$$

$$= [0{,}909] + \frac{1}{2}\{[54{,}849] + [20{,}902]\} = [0{,}909] + [37{,}876] = [38{,}785]$$

d. Compare the state trajectories of parts a, b, and c.

Penyelesaian :

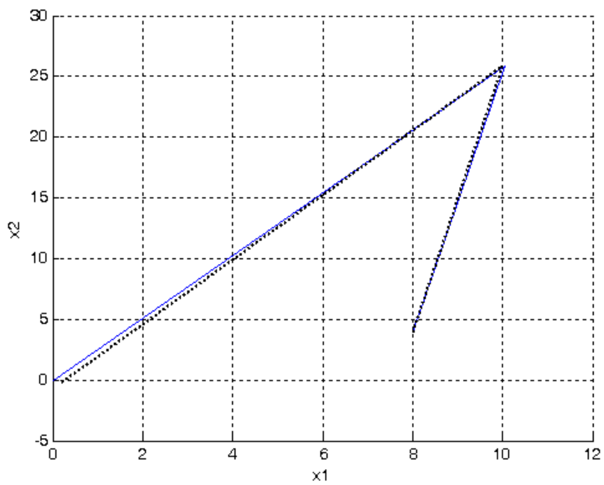
- Untuk fixed final state diperoleh:

$$x_0 = \begin{bmatrix} 8 \\ 4 \end{bmatrix}; x_1 = \begin{bmatrix} \frac{412}{41} \\ \frac{1060}{41} \end{bmatrix}; x_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Untuk free final state diperoleh :

$$x_0 = \begin{bmatrix} 8 \\ 4 \end{bmatrix}; x_1 = \begin{bmatrix} 9{,}984 \\ 25{,}832 \end{bmatrix}; x_2 = \begin{bmatrix} 0{,}191 \\ -0{,}234 \end{bmatrix}$$

Perbandingan trajectory untuk fixed dan free final state, dapat dilihat pada gambar berikut :



e. Now suppose $x_0 = [1 \quad 2]^T$. How must the controls of parts a, b, and c be modified?

Penyelesaian:

- Bagian a : Kasus *open loop control* (fixed-final state) dengan

$$x_2 = 0\,,\;\; x_0 = [1 \quad 2]^T,\;\; \text{dan}\;\; J_a = \frac{1}{2}\sum_{k=0}^{1} u_k^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} u_k$$

maka analog dengan bagian a:

$$x_k = A^k x_0 - \sum_{i=0}^{k-1} A^{k-i-1} BR^{-1} B^T \left(A^T\right)^{1-i} p_2$$

$$u_k = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \left(A^T\right)^{1-k} p_2$$

Sehingga :

$$x_2 = A^2 x_0 - \sum_{i=0}^{1} A^{1-i} BR^{-1} B^T \left(A^T\right)^{1-i} p_2$$

$$G_{0,2} = \sum_{i=0}^{1} A^{1-i} BR^{-1} B^T \left(A^T\right)^{1-i} = \left(ABR^{-1} B^T A^T\right) + \left(BR^{-1} B^T\right)$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 3 \\ \frac{1}{2} & -1 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{3}{2} \\ -\frac{3}{2} & 17 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{13}{4} & \frac{1}{2} \\ \frac{1}{2} & 19 \end{bmatrix}$$

Maka :

$$p_2 = \left(G_{0,2}\right)^{-1}\left(A^2 x_0 - x_2\right) = \begin{bmatrix} \frac{38}{123} & \frac{-1}{123} \\ \frac{-1}{123} & \frac{13}{246} \end{bmatrix} \left\{ \begin{bmatrix} \frac{7}{4} & \frac{-3}{4} \\ \frac{-9}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} \frac{38}{123} & \frac{-1}{123} \\ \frac{-1}{123} & \frac{13}{246} \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{41} \\ \frac{1}{41} \end{bmatrix}$$

$$u_0 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} (A^T) p_2 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 3 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} \frac{3}{41} \\ \frac{1}{41} \end{bmatrix} = \begin{bmatrix} \frac{-2}{41} \\ \frac{-3}{82} \end{bmatrix}$$

$$u_1 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} p_2 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{41} \\ \frac{1}{41} \end{bmatrix} = \begin{bmatrix} \frac{-4}{41} \\ \frac{-3}{41} \end{bmatrix}$$

Cek /simulasi :

$$x_1 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} x_0 + \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} u_0 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{-2}{41} \\ \frac{-3}{82} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{-11}{82} \\ \frac{-4}{41} \end{bmatrix} = \begin{bmatrix} \frac{15}{41} \\ \frac{37}{41} \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} u_1 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \frac{15}{41} \\ \frac{37}{41} \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{-4}{41} \\ \frac{-3}{41} \end{bmatrix} = \begin{bmatrix} \frac{11}{41} \\ \frac{8}{41} \end{bmatrix} + \begin{bmatrix} \frac{-11}{41} \\ \frac{-8}{41} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Bagian c : Kasus *closed loop control* (free-final state) dengan

$$J_c = 10 x_2^T x_2 + \frac{1}{2} \sum_{k=0}^{1} u_k^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} u_k \,,$$

maka analog dengan bagian a:

Kontrol Optimal :

$$u_k = -R^{-1} B^T S_{k+1} \left(I + BR^{-1} B^T S_{k+1} \right)^{-1} A x_k$$

Sehingga:

- $u_0 = -R^{-1} B^T S_1 \left(I + BR^{-1} B^T S_1 \right)^{-1} A x_0$

$$= \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 13585/901 & -5465/901 \\ -5465/901 & 2225/901 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 13585/901 & -5465/901 \\ -5465/901 & 2225/901 \end{bmatrix} \right)^{-1} \begin{bmatrix} -1/2 & 1/2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -8120/901 & 3240/901 \\ -13585/901 & 5465/901 \end{bmatrix} \begin{bmatrix} 30726/901 & -11945/901 \\ 16240/901 & -5579/901 \end{bmatrix}^{-1} \begin{bmatrix} -1/2 & 1/2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

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$$= \begin{bmatrix} -8120/901 & 3240/901 \\ -13585/901 & 5465/901 \end{bmatrix} \begin{bmatrix} -797/3578 & 11945/25046 \\ -1160/1789 & 15363/12523 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -610/12523 \\ -265/7156 \end{bmatrix} \approx \begin{bmatrix} -0{,}048 \\ -0{,}037 \end{bmatrix}$$

- $u_1 = -R^{-1} B^T S_2 \left(I + BR^{-1} B^T S_2 \right)^{-1} A x_0$

$$= \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} \right)^{-1} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -20 & -20 \\ -20 & 0 \end{bmatrix} \begin{bmatrix} \frac{41}{901} & \frac{-40}{901} \\ \frac{-40}{901} & \frac{-461}{901} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-20}{901} & \frac{-420}{901} \\ \frac{-820}{901} & \frac{800}{901} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-3580}{36941} \\ \frac{-2620}{36941} \end{bmatrix} \approx \begin{bmatrix} -0,097 \\ -0,071 \end{bmatrix}$$

- $x_1 = Ax_0 + Bu_0 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{-610}{12523} \\ \frac{-265}{7156} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{-6735}{50092} \\ \frac{-1220}{12523} \end{bmatrix} = \begin{bmatrix} \frac{18311}{50092} \\ \frac{11303}{12523} \end{bmatrix} \approx \begin{bmatrix} 0,365 \\ 0,903 \end{bmatrix}$
- $x_2 = Ax_1 + Bu_1 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \frac{18311}{50092} \\ \frac{11303}{12523} \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{-3580}{36941} \\ \frac{-2620}{36941} \end{bmatrix} = \begin{bmatrix} \frac{3843}{14312} \\ \frac{9721}{50092} \end{bmatrix} + \begin{bmatrix} \frac{-9780}{36941} \\ \frac{-7160}{36941} \end{bmatrix} \approx \begin{bmatrix} 0,00377 \\ 0,00024 \end{bmatrix}$

2.3-1 Digital Control of Harmonic Oscillator

An harmonic oscillator is described by

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\omega_n^2 x_1 + u \qquad \text{…(2.1)}$$

a. Discretize the plant using a sampling period of T.

Penyelesaian:

Sistem (2.1) dapat ditulis :

$$\dot{x}(t) = Ax(t) + Bu(t)$$

dimana : $A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix}$ dan $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Diskritisasi dari sistem diatas dengan periode sampling T adalah:

$$x_{k+1} = A^s x_k + B^s u_k$$

dimana :

- $A^s = e^{AT} = I + \sum_{n=1}^{\infty} \frac{A^n T^n}{n!}$

$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & 1\\ -\omega_n^2 & 0\end{bmatrix}T+\frac{1}{2!}\begin{bmatrix}-\omega_n^2 & 0\\ 0 & -\omega_n^2\end{bmatrix}T^2+\frac{1}{3!}\begin{bmatrix}0 & -\omega_n^2\\ \omega_n^4 & 0\end{bmatrix}T^3+\frac{1}{4!}\begin{bmatrix}\omega_n^4 & 0\\ 0 & \omega_n^4\end{bmatrix}T^4+\frac{1}{5!}\begin{bmatrix}0 & \omega_n^4\\ -\omega_n^6 & 0\end{bmatrix}T^5+\cdots$$

$$A_{11}^s=1-\frac{1}{2!}(\omega_nT)^2+\frac{1}{4!}(\omega_nT)^4+\cdots \qquad =\cos(\omega_nT)$$

$$A_{12}^s=T-\frac{1}{3!}\omega_n^2T^3+\frac{1}{5!}\omega_n^4T^5+\cdots \qquad =\frac{\sin(\omega_nT)}{\omega_n}$$

$$A_{21}^s=-\omega_n^2T+\frac{1}{3!}\omega_n^4T^3-\frac{1}{5!}\omega_n^6T^5+\cdots \qquad =-\omega_n\sin(\omega_nT)$$

$$A_{22}^s=1-\frac{1}{2!}(\omega_nT)^2+\frac{1}{4!}(\omega_nT)^4+\cdots \qquad =\cos(\omega_nT)$$

Jadi,

$$A^s=\begin{bmatrix}\cos(\omega_nT) & \dfrac{\sin(\omega_nT)}{\omega_n}\\ -\omega_n\sin(\omega_nT) & \cos(\omega_nT)\end{bmatrix}$$

- $$B^s=\int_0^T e^{A\tau}Bd\tau=\int_0^T\begin{bmatrix}\cos(\omega_n\tau) & \dfrac{\sin(\omega_n\tau)}{\omega_n}\\ -\omega_n\sin(\omega_n\tau) & \cos(\omega_n\tau)\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}d\tau$$

$$=\int_0^T\begin{bmatrix}\dfrac{\sin(\omega_n\tau)}{\omega_n}\\ \cos(\omega_n\tau)\end{bmatrix}d\tau=\left.\begin{bmatrix}\dfrac{-\cos(\omega_n\tau)}{\omega_n^2}\\ \dfrac{\sin(\omega_n\tau)}{\omega_n}\end{bmatrix}\right|_{\tau=0}^{\tau=T}=\begin{bmatrix}\dfrac{1-\cos(\omega_nT)}{\omega_n^2}\\ \dfrac{\sin(\omega_nT)}{\omega_n}\end{bmatrix}$$

Dengan demikian, diperoleh suatu sistem diskrit:

$$x_{k+1}=\begin{bmatrix}\cos(\omega_nT) & \dfrac{\sin(\omega_nT)}{\omega}\\ -\omega\sin(\omega_nT) & \cos(\omega_nT)\end{bmatrix}x_k+\begin{bmatrix}\dfrac{1-\cos(\omega_nT)}{\omega_n^2}\\ \dfrac{\sin(\omega_nT)}{\omega}\end{bmatrix}u_k \qquad \ldots(2.2)$$

b. With the discretized plant, associate a performance index of

$$J=\frac{1}{2}\left[s_1(x_N^1)^2+s_2(x_N^2)^2\right]+\frac{1}{2}\sum_{k=0}^{N-1}\left[q_1(x_k^1)^2+q_2(x_k^2)^2+ru_k^2\right]$$

where the state is $x_k=\begin{bmatrix}x_k^1 & x_k^2\end{bmatrix}^T$. Write scalar equations for a digital optimal controller.

Penyelesaian:

Diketahui sistem :

$$x_{k+1}=Ax_k+Bu_k$$

dimana : $A = \begin{bmatrix} \cos(\omega_n T) & \dfrac{\sin(\omega_n T)}{\omega_n} \\ -\omega_n \sin(\omega_n T) & \cos(\omega_n T) \end{bmatrix}$; $B = \begin{bmatrix} \dfrac{1-\cos(\omega_n T)}{\omega_n^2} \\ \dfrac{\sin(\omega_n T)}{\omega_n} \end{bmatrix}$

dan $J = \frac{1}{2} x_N^T \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} x_N + \frac{1}{2} \sum_{k=0}^{1} x_k^T \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} x_k + r u_k^2$

Oleh karena S_k simetri untuk setiap *k*, maka misalkan :

- $S = \begin{bmatrix} s_a & s_b \\ s_b & s_c \end{bmatrix}$
- $\delta = B^T S_{k+1} B + R$

$$= \begin{bmatrix} \dfrac{1-\cos(\omega_n T)}{\omega_n^2} & \dfrac{\sin(\omega_n T)}{\omega_n} \end{bmatrix} \begin{bmatrix} s_a & s_b \\ s_b & s_c \end{bmatrix} \begin{bmatrix} \dfrac{1-\cos(\omega_n T)}{\omega_n^2} \\ \dfrac{\sin(\omega_n T)}{\omega_n} \end{bmatrix} + [r]$$

$$= \frac{\left(\dfrac{1-\cos(\omega_n T) s_a}{\omega_n^2} + \dfrac{\sin(\omega_n T) s_b}{\omega} \right)(1-\cos(\omega_n T))}{\omega_n^2} + \frac{\left(\dfrac{1-\cos(\omega_n T) s_b}{\omega_n^2} + \dfrac{\sin(\omega_n T) s_c}{\omega_n} \right)\sin(\omega_n T)}{\omega_n} + r$$

maka:

$K_k = [k_1 \quad k_2] = (B^T S_{k+1} A)/\delta$

$$= \frac{1}{\delta} \left(\begin{bmatrix} \dfrac{1-\cos(\omega_n T)}{\omega_n^2} & \dfrac{\sin(\omega_n T)}{\omega_n} \end{bmatrix} \begin{bmatrix} s_a & s_b \\ s_b & s_c \end{bmatrix} \begin{bmatrix} \cos(\omega_n T) & \dfrac{\sin(\omega_n T)}{\omega_n} \\ -\omega_n \sin(\omega_n T) & \cos(\omega_n T) \end{bmatrix} \right)$$

$$k_1 = \frac{1}{\delta} \left(\left(\frac{1-\cos(\omega_n T) s_a}{\omega_n^2} + \frac{\sin(\omega_n T) s_b}{\omega_n} \right) \cos(\omega_n T) - \left(\frac{1-\cos(\omega_n T) s_b}{\omega_n^2} + \frac{\sin(\omega_n T) s_c}{\omega_n} \right) \omega_n \sin(\omega_n T) \right)$$

$$k_2 = \frac{1}{\delta} \left(\frac{\left(\dfrac{1-\cos(\omega_n T) s_a}{\omega_n^2} + \dfrac{\sin(\omega_n T) s_b}{\omega_n} \right) \sin(\omega_n T)}{\omega_n} + \left(\frac{1-\cos(\omega_n T) s_b}{\omega_n^2} + \frac{\sin(\omega_n T) s_c}{\omega_n} \right) \cos(\omega_n T) \right)$$

Matriks plant closed-loop adalah:

$$A_k^{cl} = A - BK_k = \begin{bmatrix} \cos(\omega_n T) - \frac{1-\cos(\omega_n T)k_1}{\omega_n^2} & \frac{\sin(\omega_n T)}{\omega_n} - \frac{1-\cos(\omega_n T)k_2}{\omega_n^2} \\ -\omega_n \sin(\omega_n T) - \frac{\sin(\omega_n T)k_1}{\omega_n} & \cos(\omega_n T) - \frac{\sin(\omega_n T)k_2}{\omega_n} \end{bmatrix}$$

Misalkan $A_k^{cl} = \begin{bmatrix} a_{11}^{cl} & a_{12}^{cl} \\ a_{21}^{cl} & a_{22}^{cl} \end{bmatrix}$, maka diperoleh persamaan-persamaan skalar:

$$a_{11}^{cl} = \cos(\omega_n T) - \frac{1-\cos(\omega_n T)k_1}{\omega_n^2}; \qquad a_{12}^{cl} = \frac{\sin(\omega_n T)}{\omega_n} - \frac{1-\cos(\omega_n T)k_2}{\omega_n^2};$$

$$a_{21}^{cl} = -\omega_n \sin(\omega_n T) - \frac{\sin(\omega_n T)k_1}{\omega_n}; \qquad a_{22}^{cl} = \cos(\omega_n T) - \frac{\sin(\omega_n T)k_2}{\omega_n}$$

Selanjutnya,

$$S_k = A^T S_{k+1} A_k^{cl} + Q = \begin{bmatrix} s_a & s_b \\ s_b & s_c \end{bmatrix}$$

sehingga diperoleh :

$$s_a = (\cos(\omega_n T)s_a - \omega_n \sin(\omega_n T)s_b)a_{11}^{cl} + (\cos(\omega_n T)s_b - \omega_n \sin(\omega_n T)s_c)a_{21}^{cl} + q_1$$

$$s_b = (\cos(\omega_n T)s_a - \omega_n \sin(\omega_n T)s_b)a_{12}^{cl} + (\cos(\omega_n T)s_b - \omega_n \sin(\omega_n T)s_c)a_{22}^{cl}$$

$$= \left(\frac{\sin(\omega_n T)s_a}{\omega_n} + \cos(\omega_n T)s_b\right)a_{11}^{cl} + \left(\frac{\sin(\omega_n T)s_b}{\omega_n} + \cos(\omega_n T)s_c\right)a_{21}^{cl}$$

$$s_c = \left(\frac{\sin(\omega_n T)s_a}{\omega_n} + \cos(\omega_n T)s_b\right)a_{12}^{cl} + \left(\frac{\sin(\omega_n T)s_b}{\omega_n} + \cos(\omega_n T)s_c\right)a_{22}^{cl} + q_2$$

Optimal kontrol diberikan oleh persamaan :

$$u_k = -K_k x_k$$

c. Write a MATLAB subroutine to simulate the plant dynamics, and use the time response program *lsim.m* to obtain zero-input state trajectories.

d. Write a MATLAB subroutine to compute, store the optimal control gains, and to update the control u_k given the current state . Write a MATLAB driver program to obtain tim response plots for the optimal controller.

Penyelesaian:

```
% Program  Digital Control of Harmonic Oscillator
T = 0:.05:5;
U = zeros(1,101);
A=[0 1;w^2 0]; b = [0;1] ; c= eye(2); d= zeros(2,1);
ic = [10 10];
```

```
lsim(A,b,c,dU,T,ic);
axis( [0 5 0 60] );

Function u=m(A_d,b_d,q,r,s,N,T,x0,C,S)
% Iterasi untuk Cost Kernl dan Feedback Gains
For k=N:-:1
C=cos(w*T); S= sin(w*T); w2=w^2;
Div=r + (((1-C1*s(1))/w2 + S*s(2)/w)*(1-C)/w2) + ((1-C1*s(2))/w2 + S*s(3)/w)*S/w)) ;

 % Feedback Gains
K(k,1) = (((1-C1*s(1))/w2 + S*s(2)/w)*C - ((1-C1*s(2))/w2 + S*s(3)/w)*w*S)/div;
K(k,2) = (((1-C1*s(1))/w2 + S*s(2)/w)*S/w) +((1-C1*s(2))/w2 + S*s(3)/w)*C)/div;

% Closed loop Plant Matrix
Acl(1,1) = C-(1-C*K(k,1)/w2);
Acl(1,2) = S/w-(1-C*K(k,2)/w2);
Acl(2,1) = -w*S-(S*K(k,1))/w;
Acl(2,2) = C-(S*K(k,2))/w;

% Cost Kernel Update
s(3) = (S*s(1)/w + C*s(2))*Acl(1,2)+ (S*s(2)/w+C*s(3))*Acl(2,2)+q(2);
temp=s(2);
s(2) = (C*s(1)-w*S*temp)*Acl(1,2) + (C*temp-w*S*s(3))*Acl(2,2);
s(1) =  (C*s(1)-w*S*temp)*Acl(1,1) + (C*temp-w*S*s(3))*Acl(2,1)+q(1);

% Optimal Control
x(:,1)=x0
for k=1:N
% Menghitung Optimal Control
U(k) = -K(k,:)*x(:, k);
% Plant State
x(:, k+1) = A_d* x(:, k) + b_d*u(k);
end
```

2.4-1 Steady-State Behavior.

In this problem we consider a rather unrealistic discrete system because it is simple enough to allow an analytic treatment. Thus, let the plant

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \qquad \text{…(3.1)}$$

have performance index of

$$J_0 = \frac{1}{2} x_N^T x_N + \frac{1}{2} \sum_{k=0}^{N-1} \left(x_k^T \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix} x_k + r u_k^2 \right) \qquad \text{...(3.2)}$$

a. Find the optimal steady-state (i.e., $N \to \infty$) Riccatti solution S_∞^* and show that it is positive definite. Find the optimal steady-state gain K_∞^* and determine when it is nonzero.

Penyelesaian:

- Diketahui plant : $x_{k+1} = Ax_k + Bu_k$,

dengan $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- Jika $N \to \infty$ maka $x_\infty = 0$, sehingga index performansi dapat dimodifikasi menjadi:

$$J = \frac{1}{2} \sum_{k=0}^{\infty} \left(x_k^T Q x_k + r u_k^2 \right)$$

dengan $Q = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix}$

- Persamaan Riccatti :

$$S_k = A^T \left\{ S_{k+1} - S_{k+1} B \left(B^T S_{k+1} B + R \right)^{-1} B^T S_{k+1} \right\} A + Q$$

Oleh karena $N \to \infty$ dan misalkan S_k konvergen, maka $S = S_k = S_{k+1}$.

Misalkan $S = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$, maka :

$$S = A^T \left\{ S - SB \left(B^T SB + R \right)^{-1} B^T S \right\} A + Q$$

$$S=\begin{bmatrix}0&0\\1&0\end{bmatrix}\left\{\begin{bmatrix}s_1&s_2\\s_2&s_3\end{bmatrix}-\begin{bmatrix}s_1&s_2\\s_2&s_3\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\left(\begin{bmatrix}0&1\end{bmatrix}\begin{bmatrix}s_1&s_2\\s_2&s_3\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}+r\right)^{-1}\begin{bmatrix}0&1\end{bmatrix}\begin{bmatrix}s_1&s_2\\s_2&s_3\end{bmatrix}\right\}\begin{bmatrix}0&1\\0&0\end{bmatrix}+\begin{bmatrix}q_1&q_2\\q_2&q_3\end{bmatrix}$$

$$=\begin{bmatrix}0&0\\1&0\end{bmatrix}\left\{\begin{bmatrix}s_1&s_2\\s_2&s_3\end{bmatrix}-\begin{bmatrix}s_2\\s_3\end{bmatrix}\left(\begin{bmatrix}0&1\end{bmatrix}\begin{bmatrix}s_1&s_2\\s_2&s_3\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}+r\right)^{-1}\begin{bmatrix}0&1\end{bmatrix}\begin{bmatrix}s_1&s_2\\s_2&s_3\end{bmatrix}\right\}\begin{bmatrix}0&1\\0&0\end{bmatrix}+\begin{bmatrix}q_1&q_2\\q_2&q_3\end{bmatrix}$$

$$=\begin{bmatrix}0&0\\1&0\end{bmatrix}\left\{\begin{bmatrix}s_1&s_2\\s_2&s_3\end{bmatrix}-\begin{bmatrix}s_2\\s_3\end{bmatrix}\left[r+s_3\right]^{-1}\begin{bmatrix}0&1\end{bmatrix}\begin{bmatrix}s_1&s_2\\s_2&s_3\end{bmatrix}\right\}\begin{bmatrix}0&1\\0&0\end{bmatrix}+\begin{bmatrix}q_1&q_2\\q_2&q_3\end{bmatrix}$$

$$=\begin{bmatrix}0&0\\1&0\end{bmatrix}\left\{\begin{bmatrix}s_1&s_2\\s_2&s_3\end{bmatrix}-\begin{bmatrix}\dfrac{s_2^{\,2}}{r+s_3}&\dfrac{s_2s_3}{r+s_3}\\\dfrac{s_2s_3}{r+s_3}&\dfrac{s_3^{\,2}}{r+s_3}\end{bmatrix}\right\}\begin{bmatrix}0&1\\0&0\end{bmatrix}+\begin{bmatrix}q_1&q_2\\q_2&q_3\end{bmatrix}$$

$$=\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}s_1-\dfrac{s_2^{\,2}}{r+s_3}&s_2-\dfrac{s_2s_3}{r+s_3}\\s_2-\dfrac{s_2s_3}{r+s_3}&s_3-\dfrac{s_3^{\,2}}{r+s_3}\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}+\begin{bmatrix}q_1&q_2\\q_2&q_3\end{bmatrix}$$

$$=\begin{bmatrix}q_1&q_2\\q_2&q_1+s_1-\dfrac{s_2^{\,2}}{r+s_3}\end{bmatrix}$$

Jadi, $S_\infty^*=\begin{bmatrix}q_1&q_2\\q_2&q_1+s_1-\dfrac{s_2^{\,2}}{r+s_3}\end{bmatrix}$ …(3.3)

Karena $S=\begin{bmatrix}s_1&s_2\\s_2&s_3\end{bmatrix}$, berarti :

$$s_1=q_1$$
$$s_2=q_2$$
$$s_3=q_1+s_1-\frac{s_2^{\,2}}{r+s_3}\Leftrightarrow s_3(r+s_3)=2q_1(r+s_3)-q_2^{\,2}$$
$$\Leftrightarrow s_3^{\,2}+(r-2q_1)s_3+q_2^{\,2}-2q_1r=0$$
$$\Leftrightarrow s_3=\frac{(2q_1-r)\pm\sqrt{(r-2q_1)^2-4(q_2^{\,2}-2q_1r)}}{2}$$
$$\Leftrightarrow s_3=\frac{(2q_1-r)\pm\sqrt{r^2+4(q_1^{\,2}-q_2^{\,2})+4q_1r}}{2}$$

Oleh karena:

- $r > 0$
- Q adalah matriks definit positif, berarti $q_1 > 0$, dan $\det(Q) = \left(q_1{}^2 - q_2{}^2\right) > 0$.

Dengan demikian, $\det(S) = \left(q_1{}^2 - q_2{}^2\right) - \dfrac{q_1 r}{2} \pm \dfrac{\sqrt{r^2 + 4\left(q_1{}^2 - q_2{}^2\right) + 4q_1 r}}{2} > 0$

Jadi S adalah matriks definit positif.

- $K_\infty^* = K_k = \left(B^T SB + R\right)^{-1} B^T SA$

$$= \left[\frac{1}{[r + s_3]}\right] \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dfrac{s_2}{r + s_3} \end{bmatrix}$$

Berarti $K_\infty^* \neq 0$ jika $\dfrac{s_2}{r + s_3} \neq 0$, yaitu jika $s_2 \neq 0$.

b. Find the optimal steady-state closed-loop plant and demonstrate its stability.

Penyelesaian:

Misal A_∞^{cl} adalah *steady-state closed loop plant matrix*, maka:

$$A_\infty^{cl} = A - BK_\infty^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & \dfrac{s_2}{r + s_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{s_2}{r + s_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{s_2}{r + s_3} \end{bmatrix}$$

$$x_{k+1} = \left(A - BK_\infty^*\right) x_k = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{s_2}{r + s_3} \end{bmatrix} x_k$$

c. Now the suboptimal constant feedback

$$u_k = -K_\infty^* x_k$$

is applied to the plant. Find scalar updates for the components of the suboptimal cost kernel S_k. Find the suboptimal steady-state cost kernel S_∞ and demonstrate that $S_\infty = S_\infty^*$.

Penyelesaian:

- Misalkan : $K_k = [k_1 \quad k_2]$,

maka matriks plant closed-loop adalah:

$$A_k^{cl} = A - BK_k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$$

Misalkan $A_k^{cl} = \begin{bmatrix} a_{11}^{cl} & a_{12}^{cl} \\ a_{21}^{cl} & a_{22}^{cl} \end{bmatrix}$, maka diperoleh persamaan-persamaan skalar:

$$a_{11}^{cl} = 0; \qquad a_{12}^{cl} = 1;$$
$$a_{21}^{cl} = -k_1 \qquad a_{22}^{cl} = -k_2$$

Updated cost kernel adalah:

$$S_k = A^T S A_k^{cl} + Q$$
$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \begin{bmatrix} a_{11}^{cl} & a_{12}^{cl} \\ a_{21}^{cl} & a_{22}^{cl} \end{bmatrix} + \begin{bmatrix} q_1 & q_2 \\ q_2 & q_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ s_1 a_{11}^{cl} + s_2 a_{21}^{cl} & s_1 a_{12}^{cl} + s_2 a_{22}^{cl} \end{bmatrix} + \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 \end{bmatrix}$$
$$= \begin{bmatrix} q_1 & q_2 \\ q_2 + s_1 a_{11}^{cl} + s_2 a_{21}^{cl} & q_1 + s_1 a_{12}^{cl} + s_2 a_{22}^{cl} \end{bmatrix}$$
$$= \begin{bmatrix} q_1 & q_2 \\ q_2 + q_1 a_{11}^{cl} + s_2 a_{21}^{cl} & q_1 + q_1 a_{12}^{cl} + s_2 a_{22}^{cl} \end{bmatrix}$$

- Misalkan S_∞ adalah *sub-optimal steady state cost kernel* maka :

$$S_\infty = A^T S A_\infty^{cl} + Q$$
$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -\frac{s_2}{r + s_3} \end{bmatrix} + \begin{bmatrix} q_1 & q_2 \\ q_2 & q_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & s_1 - \frac{s_2{}^2}{r + s_3} \end{bmatrix} + \begin{bmatrix} q_1 & q_2 \\ q_2 & q_2 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 + s_1 - \frac{s_2{}^2}{r + s_3} \end{bmatrix}$$

Dari persamaan (3.3), $S_\infty^* = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 + s_1 - \frac{s_2{}^2}{r + s_3} \end{bmatrix}$

Berati $S_\infty = S_\infty^*$