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From Geometry to Algebra and Vice Versa: Realistic Mathematics Education Principles for Analyzing Geometry Tasks

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Abstract. In this article we address how Realistic Mathematics Education (RME) principles, including the intertwinement and the reality principles, are used to analyze geometry tasks. To do so, we carried out three phases of a small-scale study. First we analyzed four geometry problems – considered as tasks inviting the use of problem solving and reasoning skills – theoretically in the light of the RME principles. Second, we tested two problems to 31 undergraduate students of mathematics education program and other two problems to 16 master students of primary mathematics education program. Finally, we analyzed student written work and compared these empirical to the theoretical results. We found that there are discrepancies between what we expected theoretically and what occurred empirically in terms of mathematization and of intertwinement of mathematical concepts from geometry to algebra and vice versa. We conclude that the RME principles provide a fruitful framework for analyzing geometry tasks that, for instance, are intended for assessing student problem solving and reasoning skills.

INTRODUCTION

Geometry, an indispensable topic in mathematics, is considered to be a rich area to foster student problem solving and reasoning skills ([1], [2], [3]). Problem solving skills in mathematics include abilities to understand a problem, to devise a plan, to carry out the plan, and to look back at the solution process and the solution, where these ways are not recognized previously by the problem solvers (e.g., [4], [5]); and reasoning skills cover abilities to read and do mathematical proofs [6]. The opportunity to foster the development of these skills should be exploited through, for instance, providing appropriate tasks. Such tasks often relate various concepts within the domain of mathematics, for example, between concepts of geometry and concepts of algebra. The question is how we verify if provided tasks really promote the intended skills and in particular assess student abilities in relating various mathematical concepts.

To address this question, we propose to apply two didactical principles, emerging from the theory of Realistic Mathematics Education (RME), for analyzing tasks that promote problem solving and reasoning skills [7]. According to the theory of RME, mathematics is seen as a human activity [8] and students should be active participants rather than passive receivers of ready-made mathematics in the learning and teaching process ([7], [9]). This point of departure led to two RME principles for analyzing tasks, including the reality and the intertwinement principles.

In the light of the reality principle, mathematics should start from problems that are meaningful to students, i.e. problems in rich contexts that need to be mathematized ([7], [10], [11]). Mathematization refers to an activity of translating a problem situation into the symbolic world of mathematics and vice versa, as well as reorganizing and reconstructing within the world of mathematics ([10], [11], [12], [13]). According to De Lange [12], the mathematization process, for instance in the context of geometry, is carried out by a student as follows. First, it starts with a geometry problem within the geometry world. Next, the student should identify relevant mathematics and should reorganize the problem into an algebraic model within the algebra world. The model is then solved by using algebraic procedures. Finally, the solution is reinterpreted into the initial context of the geometry problem.

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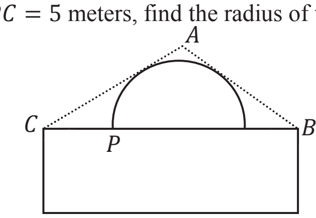
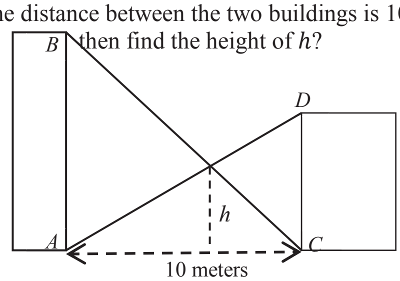
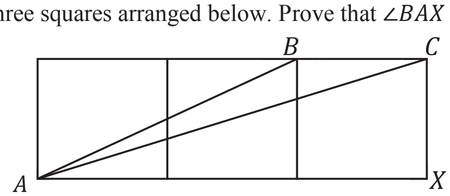
According to the intertwinement principle, mathematical content domains such as number, algebra and geometry are considered as integrated rather than as isolated curriculum chapters ([7], [10]). In the context of solving geometry problems, students are offered rich geometry problems in which they can use various mathematical concepts and procedures: not only within the domain of geometry, but also in other domains, including algebra and number [7].

METHOD

To analyze geometry tasks – promoting problem solving and reasoning skills and assessing the use of various mathematical concepts – we carried out three phases of a small-scale qualitative study involving 31 undergraduate students of mathematics education program and 16 master students of primary mathematics education program, in which both groups of students were in the first year of their academic study. First, we selected and adapted four geometry tasks, inviting the use of problem solving and reasoning skills, from relevant school mathematics competition books (e.g., [14], [15], [16]) and from a relevant book (e.g., [6]). The four tasks are presented in Table 1. Tasks 1 and 2 are intended to assess student problem solving skills and the remaining two tasks are intended to evaluate reasoning skills. For the Task 1, we expected students to apply various concepts to solve the task, including the concepts of circle, right triangle, square, and the quadratic equation. For the Task 2, we expected students to use concepts of the similarity of triangles, algebraic proportions and equations. To prove the statement in the Task 3, students were expected to use a reflection concept and the formula of Pythagoras. Finally, to prove the statement in the Task 4, we expected students to use concepts of the right triangle, algebraic expressions, and properties of an isosceles triangle.

TABLE 1. Geometry tasks used in the current study

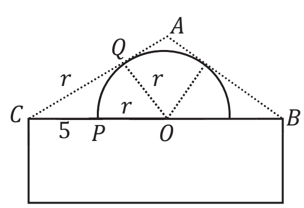
No	Geometry Tasks
1	Below is the sketch of a mosque. The dome is half of the circle that touches the right triangle ABC, where $\angle A = 90^{\circ}$. If $PC = 5$ meters, find the radius of the dome. 
2	Consider the following sketch. If $AB = 20$ meters describes the height of a building, $CD = 8$ meters describes the height of another building, the distance between the two buildings is 10 meters, *BC* and *AD* are strings, then find the height of h? 
3	Consider three squares arranged below. Prove that $\angle BAX + \angle CAX = 45^{\circ}$. 
4	If the right triangle XYZ has the lengths of its right sides x and y, the length of the hypotenuse z, and has an area of $\frac{z^2}{4}$, then prove that the triangle XYZ is isosceles.

Second, we tested Tasks 1 and 3 to undergraduate students of mathematics education program, and tested Tasks 2 and 4 to master students of primary mathematics education program. The test for each group of students lasted for about 30 minutes. Finally, we analyzed student written work in the light of the RME principles and compared these empirical to theoretical expected results. The analysis included analyzing student abilities in transforming problem situations into symbolic mathematical world from the perspective of the reality principle, and identifying the use of various related concepts in the solution process in the light of the intertwinement principle.

RESULTS AND DISCUSSION

This section presents results of the data analysis of the four tasks: Tasks 1 and 2 assessing student problem solving skills and Tasks 3 and 4 evaluating reasoning skills. We found that the four tasks are difficult for most of participated students. Of the 31 undergraduate students of mathematics education program, three students solved Task 1 and five students solved Task 3 correctly. Similarly, of the 16 master students of primary education program, all students did Task 2 incorrectly and one student solved Task 4 correctly. These results showed that both groups of students lack of abilities in dealing with geometry problems that require problem solving and reasoning skills. We below address the Task 1 in a more detail to illustrate the analysis of tasks assessing problem solving skills, and address the Task 4 to exemplify the analysis of tasks evaluating reasoning skills.

Figure 1 presents an example of expected solution process on the Task 1. From the reality principle, this solution process requires students to translate the geometry context into an algebraic quadratic equation $r^2 - 10r - 25 = 0$; to solve this equation to obtain $r = 5 \pm 5\sqrt{2}$; and to reinterpret this solution into the initial geometry context, i.e., drawing a conclusion that r must be $(5 + 5\sqrt{2})$ meters. In the light of the intertwinement principle, this task asks for students to relate various geometry concepts — including the concepts of circle, right triangle and the Pythagoras' theorem— to the concept of algebraic quadratic equations. This expectation appeared in the three students' written work, as shown for example in Figure 2. We found that student inabilities to produce correct solution process are caused by difficulties in translating the geometry context into an algebraic equation. In other words, students encountered difficulties in mathematizing the problem situation into a mathematical symbolic world (see [17]). Another quite different expected correct solution to the Task 1 is, for instance, by directly perceiving OC as the length of the hypotenuse of the isosceles right triangle OQC. So, we have the following relationship $r\sqrt{2} = r + 5 \Leftrightarrow r(\sqrt{2} - 1) = 5 \Leftrightarrow r = 5 + 5\sqrt{2}$. This second expected solution process unfortunately did not appear in student written work.



Let $OP = OQ = r$ (radius of the circle). Since OQC is a right triangle, then by applying the Pythagoras' theorem, we obtain:

$$r^2 + r^2 = (r + 5)^2$$
$$\Rightarrow r^2 - 10r - 25 = 0$$
$$\Rightarrow r = \frac{-(-10) \pm \sqrt{100 + 100}}{2}$$
$$\Rightarrow r = 5 \pm 5\sqrt{2}$$

Because the length of the radius is never negative, then the value must be $r = (5 + 5\sqrt{2})$ meters.

FIGURE 1. An example of expected solution process for the Task 1

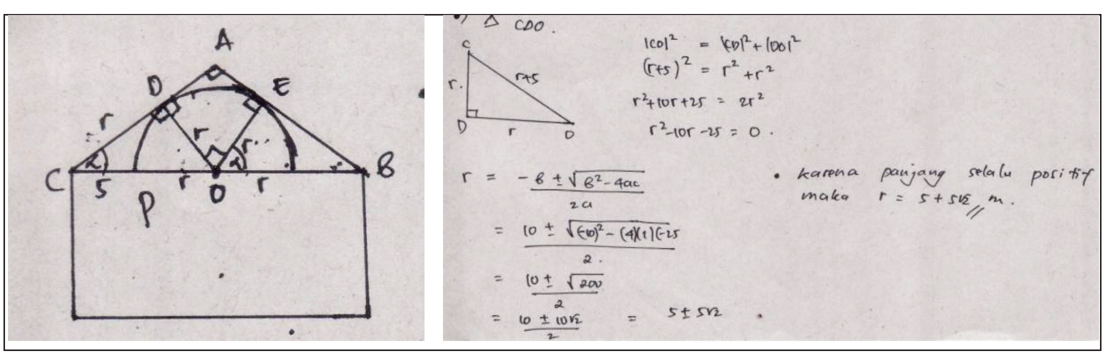
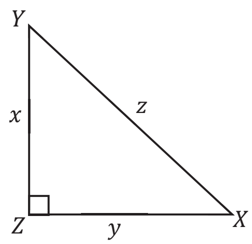


FIGURE 2. An example of student written work for the Task 1

Figure 3 shows an example of expected solution to the Task 4. From the reality principle, the task invites students to visualize information about the right triangle XYZ in the form of an appropriate figure before translating the problem situation into algebraic models $\frac{xy}{2} = \frac{z^2}{4}$ and $x^2 + y^2 = z^2$. Next, through reorganizing these algebraic models, students would arrive at the solution $x = y$. Finally, by reinterpreting the relation $x = y$ into the initial geometric context, students can conclude that the triangle XYZ is an isosceles triangle. From the perspective of the intertwinement principle, students should be able to relate between geometric concepts of the right triangle—such as the use of the Pythagoras' theorem; and algebraic concepts which make use the algebraic expressions $\frac{xy}{2} = \frac{z^2}{4}$ into $x^2 + y^2 = z^2$ to get $(x - y)^2 = 0$ and $x = y$. Figure 4 provides an example of student written work as expected. The low number of students who did the task correctly indicates that the students encountered difficulties in mathematizing the problem and in intertwining various mathematical concepts in particular. The difficulties in intertwining concepts included, for instance, relating between the expression $\frac{xy}{2} = \frac{z^2}{4}$ and $x^2 + y^2 = z^2$, and manipulating the expression $x^2 + y^2 = z^2$ into $(x - y)^2 + 2xy = z^2$.



Because the area of the right triangle is $\frac{z^2}{4}$, we have the following relation:

$$\frac{xy}{2} = \frac{z^2}{4} \Leftrightarrow 2xy = z^2.$$

By using the Pythagoras' theorem, algebraic manipulations and the previous result, we obtain:

$x^2 + y^2 = z^2.$
$\Leftrightarrow (x - y)^2 + 2xy = z^2.$
$\Leftrightarrow (x - y)^2 + z^2 = z^2.$
$\Leftrightarrow (x - y)^2 = 0.$
$\Leftrightarrow x = y.$

As $x = y$, we conclude that XYZ is an isosceles triangle.

FIGURE 3. An example of expected solution process for the Task 4

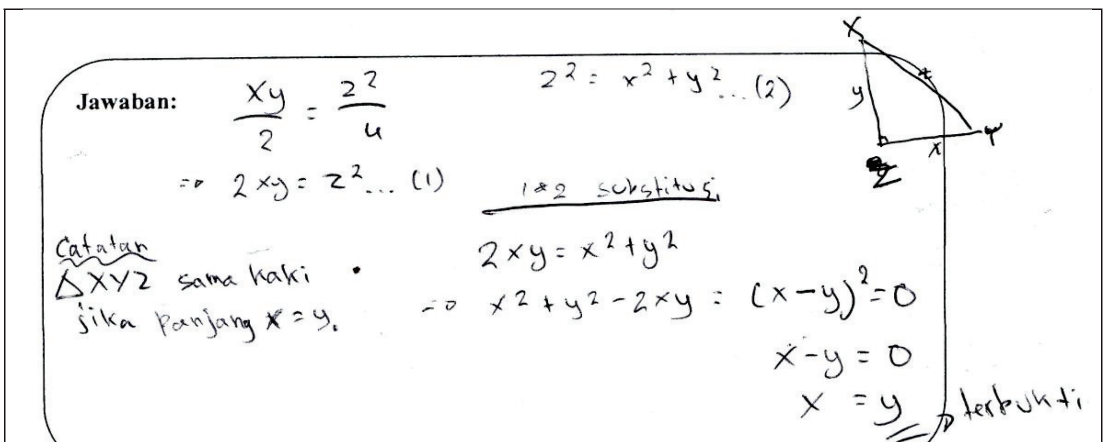


FIGURE 4. An example of student written work for the Task 4

CONCLUSION

From the results and discussion section, we draw the following two conclusions. First, both the reality and the intertwinement principles can be applied to analyze geometry tasks that promote student problem solving and reasoning skills. The reality principle is applied in the analysis to explain transformation processes from problem situations, in this case geometric contexts, to mathematical and algebraic models; from the algebraic models to solutions; and from solutions to the initial contexts of geometry problems. In other words, the reality principle explains the mathematization process of solving problems (see [12], [13]) from geometry to algebra worlds and vice versa. The intertwinement principle is used to explain relationships among mathematical domains. In the case of this study, the relationships include the relation between concepts within the domain of geometry itself and the relation between concepts of geometry and of algebra domains.

Second, student difficulties revealed in this study, including difficulties in mathematization and in relating various mathematical concepts, can be identified by applying the reality and intertwinement principles. In our view, these student difficulties show discrepancies between what we expected theoretically while designing the tasks and what occurred empirically when testing the tasks to students. To investigate this further, we consider applying the two mentioned RME principles for identifying student difficulties dealing with more various geometry tasks.

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