FINITE ELEMENT METHOD

Kartika yulianti
Jurusan Pendidikan Matematika
FPMIPA – UPI

Pendahuluan

Finite Element Method ??

Tujuan :

Mencari u sehingga $\varepsilon(u) = 0$, ε : operator differensial

Alat :

$$u(x) \approx \tilde{u}(x) = \sum_{j=1}^{N} a_j \varphi_j$$

Aproksimasi

$$u(x) = \sum_{j=1}^{\infty} a_j \varphi_j$$

$$u \in \text{crit}\ \pounds\ u\ \big|\ u \in M\ , \qquad M = \ u(x) \big| x \in \ a,b$$

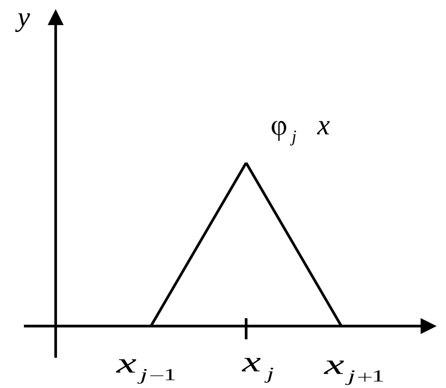
$$\tilde{u}(x) = \sum_{j=1}^{N} a_j \varphi_j$$

$$\tilde{u} \in \text{crit}\{\ \pounds\ \ u\ \big| u \in M\ \ \&\ u \in S\ \}, \qquad S \subset M$$

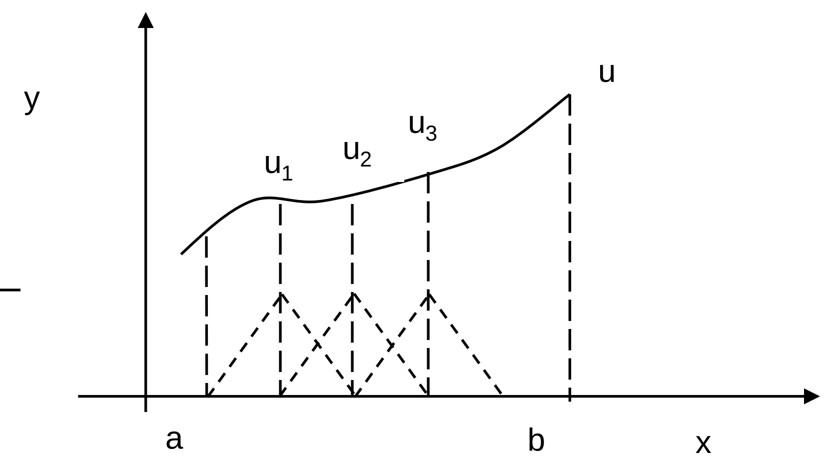
$$S_N \left\{ \sum_{j=1}^{N} a_j \varphi_j \Big| a_1, \ldots, a_N \right\}$$

<u>Basis</u>:

$$\varphi_j(x) = \begin{cases} 0, & x \notin (x_{j-1}, x_{j+1}) \\ 1 - \dfrac{|x - x_j|}{h}, & x \in (x_{j-1}, x_{j+1}) \end{cases}$$



$$a_1 = u_1$$
$$a_2 = u_2$$
$$a_N = u_N$$



Cara 1.

$$\varepsilon(u) = 0$$

$$u \in crit\{\pounds(u) \mid u \in M\}$$

$$u(x) \approx \tilde{u}(x) = \sum_{j=1}^{N} a_j \varphi_j(x)$$

$$\pounds(u) \approx \pounds(\tilde{u})$$

$$\pounds(\tilde{u}) = \int_a^b F(\tilde{u}, \partial_x \tilde{u}, x)\, dx$$

$$\pounds(\tilde{u}) = L(\vec{a}) = L(a_1, a_2, \ldots, a_N)$$

$$\text{crit}\, L(\vec{a})$$

Review

$$\text{crit}\, \pounds(u)\,?$$

$$\downarrow$$

$$\delta \pounds(u) = 0$$

$$\downarrow$$

$$u$$

Cara 2: Ritz-galerkin

$$\text{crit}\, L(\vec{a})$$

$$\frac{\partial L(\vec{a})}{\partial a_j} = 0$$

$$\left.\begin{aligned} &\frac{\partial L(\vec{a})}{\partial a_1} = 0 \\ &\vdots \\ &\frac{\partial L(\vec{a})}{\partial a_N} = 0 \end{aligned}\right\} \quad \nabla_a L = 0 \;\downarrow\; \vec{a} \;\downarrow\; \tilde{u}$$

$$\varepsilon(u) = 0$$

$$\varepsilon(\tilde{u}) = \varepsilon\left(\sum_{j=1}^{N} a_j \varphi_j\right) \approx 0$$

$$\left\langle \varepsilon\left(\sum_{j=1}^{N} a_j \varphi_j\right), \varphi_k \right\rangle = 0, \qquad k = 1,\ldots,N$$

$$\frac{\partial L(\vec{a})}{\partial a_j} = \frac{\partial \pounds\ \ \tilde{u}(\vec{a})\ }{\partial a_j} = \left\langle \partial \pounds\ \ \tilde{u}\ \ , \frac{\partial \tilde{u}}{\partial a_j} \right\rangle = \left\langle \partial \pounds\ \ \tilde{u}\ \ , \varphi_j \right\rangle = 0$$

Contoh

$$-\partial_x^2 u = x, \quad u'(0) = 0, \; u(1) = 1$$

Misal $\tilde{u} = \sum_{j=1}^{N} a_j \varphi_j$

Cara 1

$$\partial_x^2 u + x = 0$$

$$\pounds(u) = \int_0^1 -\frac{1}{2}(\partial_x u)^2 + xu \, dx$$

$$\pounds(\tilde{u}) = \int_0^1 -\frac{1}{2}\left(\partial_x \sum_{j=1}^{N} a_j \varphi_j\right)^2 + x \sum_{j=1}^{N} a_j \varphi_j \, dx$$

$$L(\vec{a}) = -\left[P_{jk}\right]\vec{a}.\vec{a} + \left[Q_k\right]\vec{a}$$

$$P_{jk} = \int_0^1 (\partial_x \varphi_j)(\partial_x \varphi_k)\, dx$$

$$\nabla L(\vec{a}) = 0 \rightarrow -\left[P_{jk}\right]\vec{a} + \left[Q_k\right] = 0$$

$$Q_k = \int_0^1 x(\varphi_k)\, dx$$

Cara 2

$$\left\langle -\partial_x^2 \left(\sum_{j=1}^{N} a_j \varphi_j \right) - x,\ \varphi_k \right\rangle = 0$$

$$\int_0^1 -\partial_x^2 \sum_{j=1}^{N} a_j \varphi_j\ \varphi_k - x\,\varphi_k\, dx = 0$$

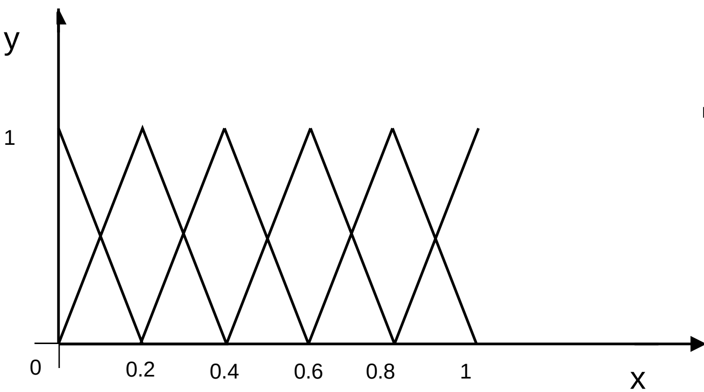
$$\varphi_j \partial_x \sum_{j=1}^{N} a_k \varphi_k \Bigg|_0^1 + \int_0^1 \sum_{j=1}^{N} a_j \partial_x \varphi_j\ \partial_x \varphi_k - x\,\varphi_k\, dx = 0$$

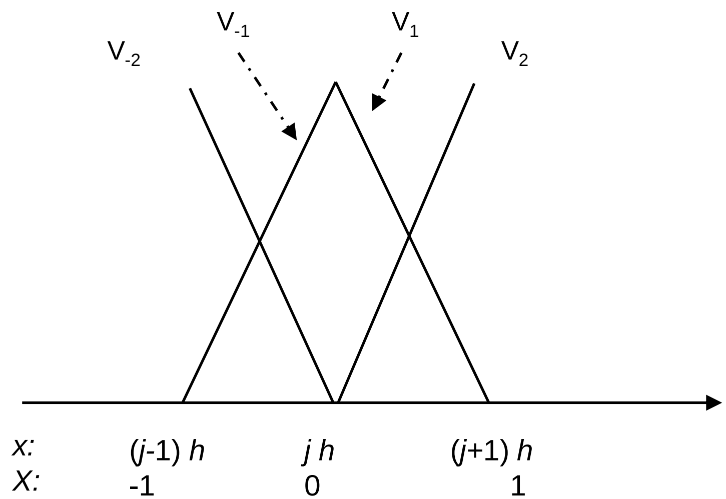
$$\partial_x \tilde{u}(0) = 0 \quad \text{dan } \varphi_j(1) = 0, \text{ untuk } j = 0,1,\ldots,N-1$$

$$\int_0^1 \sum_{j=1}^{N} a_j \partial_x \varphi_j\ \partial_x \varphi_k - x\,\varphi_k\, dx = 0 \qquad , k = 1, 2, \ldots, N\text{-}1 \quad \text{dan} \qquad \tilde{u}(1) = 1$$

$$\left[P_{jk} \right] \left[\vec{a} \right] - \left[Q_k \right] = 0$$

$$\text{Basis}\ :\quad \varphi_j(x)=\begin{cases}0, & x\notin(x_{j-1},x_{j+1})\\ 1-\dfrac{|x-x_j|}{h}, & x\in(x_{j-1},x_{j+1})\end{cases}$$





$$X=\frac{x}{h}-k$$

$V_{-1} = 1 + X$ $\quad$ $V_1 = 1 - X$
$V_{-2} = -X$ $\quad$ $V_2 = X$

$-1\le X\le 0$	$0\le X\le 1$
$\tilde u(X)=-Xa_{-1}+(1+X)a_0$	$\tilde u(X)=(1-X)a_0+Xa_1$
$\partial_x\tilde u(x)=(-a_{-1}+a_0)\frac{1}{h}$	$\partial_x\tilde u(x)=(-a_0+a_1)\frac{1}{h}$
$\varphi(X)=1+X$	$\varphi(X)=1-X$
$\partial_x\varphi(x)=\frac{1}{h}$	$\partial_x\varphi(x)=-\frac{1}{h}$

$$\int_0^1 \sum_{j=1}^{N} a_j \partial_x \varphi_j \, \partial_x \varphi_k \, dx = \int_{x_{j-1}}^{x_{j+1}} \partial_x \tilde{u}(x) \, \partial_x \varphi_k(x) dx$$

$$= \frac{1}{h} \int_{-1}^{1} \frac{d\tilde{u}(X)}{dX} \frac{d\varphi_j(X)}{dX} dX$$

$$= \frac{1}{h} \left(-a_{j+1} + 2a_j - a_{j-1} \right)$$

$$\int_0^1 x \varphi_j(x) dx = h^2 \left[\int_{j-1}^{j} (1+X)(X+j) dX + \int_{j}^{j+1} (1-X)(X+j) dX \right] = h^2 j$$

untuk $j = 1, 2, \ldots, N\text{-}1$

$$\frac{1}{h} \left(a_{j+1} + a_j \right) = \frac{1}{6} h^2, \quad \text{untuk } j = 0$$

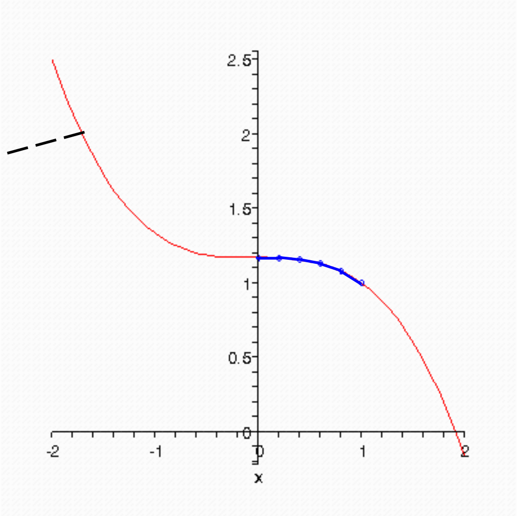
Matriks discretisasi:

$$\frac{1}{h^2} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_o \\ a_1 \\ \vdots \\ a_{N-1} \\ a_N \end{bmatrix} = \begin{bmatrix} \frac{h}{6} \\ h \\ \vdots \\ (N-1)h \\ \frac{1}{h^2} \end{bmatrix}$$

$$kogef := \begin{bmatrix} 1.166666667 \\ 1.165333333 \\ 1.156000000 \\ 1.130666667 \\ 1.081333333 \\ 1.000000000 \end{bmatrix}$$

Solusi eksak

$$u(x) = -\frac{1}{6}x^3 + \frac{7}{6}$$



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