1. Calculate the first variation of the following functionals:

a. $A(u)=\int_0^1 x\,u(x)^2+\left(\frac{\partial u(x)}{\partial x}\right)^2 dx$

Penyelesaian:

$$\frac{\partial}{\partial\varepsilon}A(u+\varepsilon\eta)\Big|_{\varepsilon=0}=\frac{\partial}{\partial\varepsilon}\int_0^1 x\,(u+\varepsilon\eta)^2+\left(\frac{\partial(u+\varepsilon\eta)}{\partial x}\right)^2 dx\Big|_{\varepsilon=0}$$

$$=\frac{\partial}{\partial\varepsilon}\int_0^1 x\,(u^2+2u\varepsilon\eta+\varepsilon^2\eta^2)+(\partial_x u+\varepsilon\partial_x\eta)^2 dx\Big|_{\varepsilon=0}$$

$$=\int_0^1 2u\eta x\,dx+\int_0^1 2\partial_x u\;\partial_x\eta\,dx$$

$$=\int_0^1 2u\eta x\,dx+2\partial_x u\,\eta-\int_0^1 2\partial_{xx}u\;\eta\,dx$$

$$=2\partial_x u\,\eta\Big|_0^1+\int_0^1(2ux-2\partial_{xx}u)\,\eta\,dx \qquad \Leftrightarrow \text{Bcs}+\langle\delta A,\eta\rangle$$

Maka turunan variasi dari $A(u(x))$ adalah $\delta A(u(x))=2x\,u(x)-2\partial_{xx}u(x)$.

b. $A(u)=\int \sin x\;u(x)^2+x^3\left(\frac{\partial u(x)}{\partial x}\right)^2 dx$

Penyelesaian:

$$\frac{\partial}{\partial\varepsilon}A(u+\varepsilon\eta)\Big|_{\varepsilon=0}=\frac{\partial}{\partial\varepsilon}\int_0^1 \sin x\,(u+\varepsilon\eta)^2+x^3\left(\frac{\partial(u+\varepsilon\eta)}{\partial x}\right)^2 dx\Big|_{\varepsilon=0}$$

$$=\frac{\partial}{\partial\varepsilon}\int_0^1 \sin x\,(u^2+2u\varepsilon\eta+\varepsilon^2\eta^2)+x^3(\partial_x u+\varepsilon\partial_x\eta)^2 dx\Big|_{\varepsilon=0}$$

$$=\int_0^1 2u\eta\sin x\,dx+\int_0^1 2x^3\partial_x u\;\partial_x\eta\,dx$$

$$=\int_0^1 2u\eta\sin x\,dx+2x^3\partial_x u\,\eta\Big|_0^1-\int_0^1(6x^2\partial_x u+2x^3\partial_{xx}u)\,\eta\,dx$$

$$=2x^3\partial_x u\,\eta\Big|_0^1+\int_0^1(2u\sin x-6x^2\partial_x u-2x^3\partial_{xx}u)\,\eta\,dx \qquad \Leftrightarrow \text{Bcs}+\langle\delta A,\eta\rangle$$

Maka turunan variasi dari $A[u(x)]$ adalah

$\delta A[u(x)] = 2u(x)\sin x - 6x^2\partial_x u(x) - 2x^3\partial_{xx}u(x)$.

c. $A(u) = \int u(x)^2 + \left(\frac{\partial u(x)}{\partial x}\right)^2 dx$

Penyelesaian:

$$\frac{\partial}{\partial\varepsilon}A[u+\varepsilon\eta]\Big|_{\varepsilon=0} = \frac{\partial}{\partial\varepsilon}\int (u+\varepsilon\eta)^2 + \left(\frac{\partial(u+\varepsilon\eta)}{\partial x}\right)^2 dx\Big|_{\varepsilon=0}$$

$$= \int \frac{\partial}{\partial\varepsilon}(u+\varepsilon\eta)^2\Big|_{\varepsilon=0} + \frac{\partial}{\partial\varepsilon}(\partial_x u+\varepsilon\partial_x\eta)^2\Big|_{\varepsilon=0} dx$$

$$= \int 2u\eta\,dx + \int 2\partial_x u\,\partial_x\eta\,dx$$

$$= \int 2u\eta\,dx + 2(\partial_x u\,\eta)\Big|_{x=D} - \int 2\partial_{xx}u\,\eta\,dx$$

$$= 2\partial_x u\,\eta\Big|_{x=D} + \int (2u - 2\partial_{xx}u)\eta\,dx$$

$$\Leftrightarrow \text{Bcs} + \langle\delta A,\eta\rangle$$

Turunan variasi dari $A[u(x)]$ adalah $\delta A[u(x)] = 2u(x) - 2\partial_{xx}u(x)$.

d. $A(u) = \int_0^1 \sin u(x) + \left(\frac{\partial^2 u(x)}{\partial x^2}\right)^2 dx$

Penyelesaian:

$$\frac{\partial}{\partial\varepsilon}A[u+\varepsilon\eta]\Big|_{\varepsilon=0} = \frac{\partial}{\partial\varepsilon}\int_0^1 \sin(u+\varepsilon\eta) + \left(\frac{\partial^2(u+\varepsilon\eta)}{\partial x^2}\right)^2 dx\Big|_{\varepsilon=0}$$

$$= \frac{\partial}{\partial\varepsilon}\int_0^1 \sin(u+\varepsilon\eta) + (\partial_x(\partial_x u+\varepsilon\partial_x\eta))^2 dx\Big|_{\varepsilon=0}$$

$$= \frac{\partial}{\partial\varepsilon}\int_0^1 \sin(u+\varepsilon\eta) + (\partial_{xx}u+\varepsilon\partial_{xx}\eta)^2 dx\Big|_{\varepsilon=0}$$

$$= \int_0^1 \cos(u)\eta\,dx + \int_0^1 2(\partial_{xx}u)(\partial_{xx}\eta)dx$$

$$= \int_0^1 \cos(u)\eta\, dx + 2\left[\partial_{xx} u\right]\left[\partial_x \eta\right]\Big|_0^1 - \int_0^1 2\left[\partial_{xxx} u\right]\left[\partial_x \eta\right] dx$$

$$= 2\left[\partial_{xx} u\right]\left[\partial_x \eta\right]\Big|_0^1 - 2\left[\partial_{xxx} u\, \eta\right]\Big|_0^1 + \int_0^1 \cos(u)\eta\, dx + \int_0^1 2\left[\partial_x^{(iv)} u\, \eta\right] dx$$

$$= 2\left[\partial_{xx} u\right]\left[\partial_x \eta\right]\Big|_0^1 - 2\left[\partial_{xxx} u\, \eta\right]\Big|_0^1 + \int_0^1 \left[\cos(u) + 2\left[\partial_x^{(iv)} u\right]\right]\eta\, dx$$

Maka turunan variasi dari $A\left[u(x)\right]$ adalah $\delta A\left[u(x)\right] = \cos(u(x)) + 2\partial_x^{(iv)} u(x)$.

e. $A(u) = \int_0^1 u(x)^4 + \left(\frac{\partial u(x)}{\partial x}\right)^7 dx$

Penyelesaian:

$$\frac{\partial}{\partial \varepsilon} A\left[u + \varepsilon\eta\right]\Big|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon}\int_0^1 \left[u + \varepsilon\eta\right]^4 + \left(\frac{\partial\left[u + \varepsilon\eta\right]}{\partial x}\right)^7 dx\Big|_{\varepsilon=0}$$

$$= \int_0^1 \frac{\partial}{\partial \varepsilon}\left[u + \varepsilon\eta\right]^4\Big|_{\varepsilon=0} + \frac{\partial}{\partial \varepsilon}\left[\partial_x u + \varepsilon\partial_x \eta\right]^7\Big|_{\varepsilon=0} dx$$

$$= \int_0^1 4u^3\eta\, dx + \int_0^1 7\left[\partial_x u\right]^6 \partial_x \eta\, dx$$

$$= \int_0^1 4u^3\eta\, dx + 7\left[\partial_x u\right]^6 \eta\Big|_0^1 - \int_0^1 42\left[\partial_x u\right]^5 \partial_{xx} u\, \eta\, dx$$

$$= 7\left[\partial_x u\right]^6 \eta\Big|_0^1 + \int_0^1 \left[4u^3 - 42\left[\partial_x u\right]^5 \partial_{xx} u\right]\eta\, dx$$

$$\Leftrightarrow \text{Bcs} + \langle \delta A, \eta\rangle$$

Turunan variasi dari $A\left[u(x)\right]$ adalah $\delta A\left[u(x)\right] = 4\left[u(x)\right]^3 - 42\left[\partial_x u(x)\right]^5 \partial_{xx} u(x)$.

f. $A(u) = \int_0^1 n(x)\left(1 + \frac{\partial u(x)^2}{\partial x}\right)^{\frac{1}{2}} dx$

Penyelesaian:

Persamaan untuk turunan variasi adalah

$$\delta A(u) = \frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\frac{\partial L}{\partial \dot{u}}$$

dimana L adalah *integrand* dari *functional* dan $\dot{u} = \dfrac{\partial u(x)}{\partial x}$. Maka turunan variasi untuk soal no 1.f adalah

$$\delta A(u(x)) = \frac{\partial}{\partial x}\left[\frac{n(x)(\partial_x u(x))}{\sqrt{1+(\partial_x u(x))^2}}\right]$$

$$= \frac{(\partial_x n(x))(\partial_x u(x))}{\sqrt{1+(\partial_x u(x))^2}} + \frac{n(x)(\partial_{xx} u(x))}{\sqrt{1+(\partial_x u(x))^2}} - \frac{n(x)(\partial_x u(x))^2(\partial_{xx} u(x))}{(1+(\partial_x u(x))^2)^{\frac{3}{2}}}$$

g. $A(q) = \int \frac{1}{2}\dot{q}(t)^2 - \frac{1}{2}q(t)^2 + q(t)^3 dt$

Penyelesaian:

$$\frac{\partial}{\partial \varepsilon} A(q+\varepsilon\eta)\Big|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon}\int \frac{1}{2}\left(\frac{\partial(q+\varepsilon\eta)}{\partial t}\right)^2 - \frac{1}{2}(q+\varepsilon\eta)^2 + (q+\varepsilon\eta)^3 dt\Big|_{\varepsilon=0}$$

$$= \int \frac{\partial}{\partial \varepsilon}\left(\frac{1}{2}(\dot{q}(t)+\varepsilon\dot{\eta}(t))^2\right)\Big|_{\varepsilon=0} - \frac{\partial}{\partial \varepsilon}\left(\frac{1}{2}((q+\varepsilon\eta)(t))^2\right)\Big|_{\varepsilon=0} + \frac{\partial}{\partial \varepsilon}((q+\varepsilon\eta)(t))^3\Big|_{\varepsilon=0} dt$$

$$= \int \dot{q}(t)\dot{\eta}(t)\,dt - \int q(t)\eta(t)\,dt + \int 3(q(t))^2\eta(t)\,dt$$

$$= \dot{q}(t)\eta(t)\big|_{x=D} - \int \ddot{q}(t)\eta(t)\,dt - \int q(t)\eta(t)\,dt + \int 3(q(t))^2\eta(t)\,dt$$

$$= \dot{q}(t)\eta(t)\big|_{x=D} + \int (3(q(t))^2 - \ddot{q}(t) - q(t))\eta(t)\,dt$$

$$\Leftrightarrow \text{Bcs} + \langle \delta A, \eta \rangle$$

Turunan variasi dari $A(q)$ adalah $\delta A(q) = 3(q(t))^2 - \ddot{q}(t) - q(t)$

h. $A(u) = \int \frac{1}{2}\partial_x u(x)^2 + x^3 \sin u(x) + u(x)^5 dx$

Penyelesaian:

$$\frac{\partial}{\partial\varepsilon}A(u+\varepsilon\eta)\Big|_{\varepsilon=0} = \frac{\partial}{\partial\varepsilon}\int \frac{1}{2}\left(\frac{\partial(u+\varepsilon\eta)}{\partial x}\right)^2 + x^3\sin(u+\varepsilon\eta) + (u+\varepsilon\eta)^5 dx\Big|_{\varepsilon=0}$$

$$= \int \frac{1}{2}\frac{\partial}{\partial\varepsilon}(\partial_x u + \varepsilon\partial_x\eta)^2\Big|_{\varepsilon=0} + \frac{\partial}{\partial\varepsilon}x^3\sin(u+\varepsilon\eta)\Big|_{\varepsilon=0} + \frac{\partial}{\partial\varepsilon}(u+\varepsilon\eta)^5\Big|_{\varepsilon=0} dx$$

$$= \int(\partial_x u)(\partial_x\eta)dx + \int x^3\cos(u)\eta\, dx + \int 5u^4\eta\, dx$$

$$= (\partial_x u\,\eta)\big|_{x=D} - \int(\partial_{xx}u\,\eta)\,dx + \int(x^3\cos(u) + 5u^4)\eta\, dx$$

$$= (\partial_x u\,\eta)\big|_{x=D} + \int(x^3\cos(u) + 5u^4 - \partial_{xx}u)\eta\, dx$$

$$\Leftrightarrow \text{Bcs} + \langle\delta A, \eta\rangle$$

Turunan variasi dari $A(u(x))$ adalah $\delta A(u(x)) = x^3\cos(u(x)) + 5(u(x))^4 - \partial_{xx}u(x)$

2. *Light rays, Fermat's principle*.

According to Fermat, the trajectory of a light ray between two points is such that the time required time is as small as possible. The propagation speed of light depends on material properties, which is expressed by c_o/n where co is the speed in vacuum (which is maximal), and $n>1$ is the so called index of refraction, characteristic for the material.

For trajectories, for simplicity described as graphs of functions $x \to y(x)$. The total time between points is

$$\int_D n(x,y)\sqrt{1+y_x^2}\,dx \qquad \ldots(*)$$

this is also often called the optical pathlength. Note that this functional can also be given very different interpretations, depending on the meaning of n.

a. Write down the Euler-Lagrange equations.

b. Determine the optimal trajectory in case n does not depend on x explicitly. Then use 'energy conservation' to study trajectories.

c. Consider the special cases $n = y$ and $n = \frac{1}{y}$ for which the trajectories can be expressed explicitly.

<u>Penyelesaian</u>:

a. Misalkan $A(y) = \int_D n(x,y)\sqrt{1+y_x^2}\,dx$

Turunan variasi dari $A(y)$:

$$\frac{\partial}{\partial\varepsilon}A(y+\varepsilon\eta)\Big|_{\varepsilon=0} = \frac{\partial}{\partial\varepsilon}\int_D n(x,y+\varepsilon\eta)\sqrt{1+\left(\frac{\partial(y+\varepsilon\eta)}{\partial x}\right)^2}\,dx\,\Bigg|_{\varepsilon=0}$$

$$= \int_D \frac{\partial(n(x,y+\varepsilon\eta))}{\partial y}\eta\sqrt{1+(\partial_x y+\varepsilon\partial_x\eta)^2}\,\Big|_{\varepsilon=0}dx +$$

$$\int_D n(x,y+\varepsilon\eta)\frac{(\partial_x y)(\partial_x\eta)}{\sqrt{1+(\partial_x y+\varepsilon\partial_{x\eta})^2}}\Bigg|_{\varepsilon=0}dx$$

$$= \int_D\left(\frac{\partial n(x,y)}{\partial y}\sqrt{1+(\partial_x y)^2}\right)\eta\,dx + \int_D n(x,y)\frac{(\partial_x y)}{\sqrt{1+(\partial_x y)^2}}(\partial_x\eta)\,dx$$

$$= \int_D\left(\frac{\partial n(x,y)}{\partial y}\sqrt{1+(\partial_x y)^2}\right)\eta\,dx + \frac{n(x,y)(\partial_x y)}{\sqrt{1+(\partial_x y)^2}}\eta\,\Bigg|_D - \int_D\frac{\partial}{\partial x}\left(\frac{n(x,y)(\partial_x y)}{\sqrt{1+(\partial_x y)^2}}\right)\eta\,dx$$

$$= \frac{n(x,y)(\partial_x y)}{\sqrt{1+(\partial_x y)^2}}\eta\,\Bigg|_D + \int_D\left(\frac{\partial n(x,y)}{\partial y}\sqrt{1+(\partial_x y)^2} - \frac{\partial}{\partial x}\left(\frac{n(x,y)(\partial_x y)}{\sqrt{1+(\partial_x y)^2}}\right)\right)\eta\,dx$$

Persamaan Euler-Lagrange :

$$\frac{\partial n(x,y)}{\partial y}\sqrt{1+(\partial_x y)^2} - \frac{\partial}{\partial x}\left(\frac{n(x,y)(\partial_x y)}{\sqrt{1+(\partial_x y)^2}}\right) = 0$$

$$\Leftrightarrow \partial_y n\sqrt{1+(\partial_x y)^2} - \frac{(\partial_x n)(\partial_x y) + (\partial_y n)(\partial_x y)^2}{\sqrt{1+(\partial_x y)^2}} - n\frac{\partial_{xx}y}{\sqrt{1+(\partial_x y)^2}} + n\frac{(\partial_x y)^2(\partial_{xx}y)}{(1+(\partial_x y)^2)^{\frac{3}{2}}} = 0$$

Persamaan terakhir disederhanakan menjadi:

$$\partial_y n(1+(\partial_x y)^2) - n\,\partial_{xx}y - \partial_x n\,\partial_x y(1+(\partial_x y)^2) = 0$$

b. Dalam kasus $n(x,y)$ tidak begantung secara eksplisit pada x, maka $\frac{\partial n(x,y)}{\partial x} = 0$,

sehingga persamaan Euler-Lagrange menjadi

$$\partial_y n\sqrt{1+(\partial_x y)^2} - \frac{(\partial_y n)(\partial_x y)^2}{\sqrt{1+(\partial_x y)^2}} - n\frac{\partial_{xx} y}{\sqrt{1+(\partial_x y)^2}} + n\frac{(\partial_x y)^2(\partial_{xx} y)}{\left(1+(\partial_x y)^2\right)^{\frac{3}{2}}} = 0$$

$$\Leftrightarrow \partial_y n\ \left[1+(\partial_x y)^2\right] - n\ \partial_{xx} y = 0$$

$$\Leftrightarrow \partial_y n + \partial_y n\ (\partial_x y)^2 - n\ \partial_{xx} y = 0$$

Trajectory (solusi) dari persamaan diferensial di atas, berupa kurva kuadratik.

c. – Untuk kasus $n = y(x)$

Berdasarkan no 8.a maka persamaan Euler-Lagrange untuk kasus ini adalah:

$$\sqrt{1+(\partial_x y)^2} - \frac{(\partial_x y)^2}{\sqrt{1+(\partial_x y)^2}} - \frac{y(\partial_{xx} y)}{\sqrt{1+(\partial_x y)^2}} + \frac{y(\partial_x y)^2(\partial_{xx} y)}{\left(1+(\partial_x y)^2\right)^{\frac{3}{2}}} = 0$$

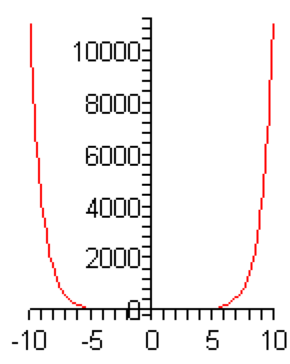
Dengan menggunakan program Maple (terdapat pada lampiran) solusi dari persamaan differensial tersebut adalah

$$y(x) = \frac{1+(e^{c_1 x})^2(e^{c_1 c_2})^2}{2c_1 e^{c_1 x} e^{c_1 c_2}},\ \text{dimana } c_1 \text{ dan } c_2 \text{ adalah konstanta}$$

Dengan mengambil nilai awal $y(0) = 1$ dan $y_x(0) = 0$ diperoleh solusi :

$$y(x) = \frac{1+(e^{-x})^2}{2e^{-x}}$$

dan gambar solusi dari persamaan diferensial tersebut (*trajectory*) adalah



$\therefore$ *Light Rays* di dalam medium dengan fungsi $n(x, y) = y(x)$ adalah berupa kurva kuadratik.

- Untuk kasus $n = \dfrac{1}{y(x)}$

Berdasarkan no 8.a maka persamaan Euler-Lagrange untuk kasus ini :

$$\frac{1}{y^2}\sqrt{1+(\partial_x y)^2} + \frac{(\partial_x y)^2}{y^2\sqrt{1+(\partial_x y)^2}} - \frac{(\partial_{xx} y)}{y\sqrt{1+(\partial_x y)^2}} + \frac{(\partial_x y)^2(\partial_{xx} y)}{y(1+(\partial_x y)^2)^{\frac{3}{2}}} = 0$$

Dengan menggunakan program Maple (terdapat pada lampiran) solusi dari persamaan differensial tersebut adalah

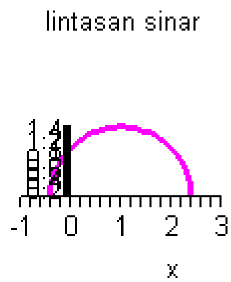
$$y_1(x) = \sqrt{-x^2 - 2c_1 x + 2c_2}\,,\ \text{dan}\ y_2(x) = -\sqrt{-x^2 - 2c_1 x + 2c_2}\,,$$

c_1 dan c_2 adalah konstanta.

Dengan mengambil nilai awal $y(0) = 0$ dan $y_x(0) = 1$ diperoleh solusi:

$$y(x) = \sqrt{-x^2 + 2x + 1}$$

Gambar trajectory dari persamaan diferensial tersebut adalah



Light rays di dalam medium dengan $n(x,y) = \dfrac{1}{y(x)}$ berupa kurva parabola (kuadratik).

3. Boussinesq type of equations.

   Surface waves (in one horizontal direction x) that decay at infinity ($|x| \to \pm\infty$) can be described in terms of the wave height $p(x,t)$ and a velocity $u(x,t)$ in the following form (a Hamiltonian system):

$$\begin{aligned} \partial_t u &= \partial_x \left(\delta_p H(u,p)\right), \\ \partial_t p &= -\partial_x \left(\delta_u H(u,p)\right) \end{aligned} \qquad \ldots\text{(i)}$$

   For a suitable functional (the Hamiltonian) $H(u,p)$.

   a. Describe the equations in full detail when the Hamiltonian is given by the following functional

$$H(u,p)=\int\left\{\frac{1}{2}gp^2+\frac{1}{2}(h+p)\left(u^2-\frac{1}{3}u_x{}^2\right)\right\}dx$$

(This set of equations are the linearized equations).

b. In another case (shallow water, no dispersion, but nonlinear), the equations are of the form

$$\partial_t u=\partial_x\left\{gp+\frac{1}{2}u^2\right\},$$
$$\partial_t p=-\partial_x\{(hu+\beta pu)\}$$

Where β is constant. Determine the value of β such that this system of equations is a Hamiltonian system of the form (3,4) given above.

c. Show that the equations have the horizontal momentum as constant of the motion

$$\int u(x)p(x)dx$$

Penyelesaian:

a. Hamiltonian: $H(u,p)=\int\left\{\frac{1}{2}gp^2+\frac{1}{2}(h+p)\left(u^2-\frac{1}{3}u_x{}^2\right)\right\}dx$

- Akan ditentukan $\delta_u H(u,p)$

$$\frac{\partial}{\partial\varepsilon}H(u+\varepsilon\eta,p)\Big|_{\varepsilon=0}=\frac{\partial}{\partial\varepsilon}\int\frac{1}{2}gp^2+\frac{1}{2}(h+p)\left[(u+\varepsilon\eta)^2-\frac{1}{3}\left(\frac{\partial(u+\varepsilon\eta)}{\partial x}\right)^2\right]dx\Bigg|_{\varepsilon=0}$$

$$=\int(h+p)u\eta\,dx-\int\frac{1}{3}(h+p)(\partial_x u)(\partial_x \eta)dx$$

$$=\int(h+p)u\eta\,dx-\frac{1}{3}(h+p)(\partial_x u)\eta\Big|_{x=a}^{b}+\int\frac{1}{3}\left[(\partial_x h+\partial_x p)(\partial_x u)+(h+p)(\partial_{xx}u)\right]\eta\,dx$$

$$=-\frac{1}{3}(h+p)(\partial_x u)\eta\Big|_{x=a}^{b}+\int\left[(h+p)u+\frac{1}{3}(\partial_x h+\partial_x p)(\partial_x u)+\frac{1}{3}(h+p)(\partial_{xx}u)\right]\eta\,dx$$

maka $\delta_u H(u,p)=(h+p)u+\frac{1}{3}(\partial_x h+\partial_x p)(\partial_x u)+\frac{1}{3}(h+p)(\partial_{xx}u)$

- Akan ditentukan $\delta_p H(u,p)$

$$\frac{\partial}{\partial \varepsilon} H(u, p+\varepsilon\eta)\Big|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} \int \frac{1}{2} g(p+\varepsilon\eta)^2 + \frac{1}{2}(h+p+\varepsilon\eta)\left(u^2 - \frac{1}{3}(\partial_x u)^2\right) dx \Big|_{\varepsilon=0}$$

$$= \int gp\eta \, dx - \int \frac{1}{2}\eta\left(u^2 - \frac{1}{3}(\partial_x u)^2\right) dx$$

$$= \int \left[gp + \frac{1}{2}\left(u^2 - \frac{1}{3}(\partial_x u)^2\right)\right]\eta \, dx$$

Maka $\delta_p H(u, p) = gp + \frac{1}{2}\left(u^2 - \frac{1}{3}(\partial_x u)^2\right)$

Sehingga sistem (i) menjadi

$$\partial_t u = \partial_x \left\{ gp + \frac{1}{2}\left(u^2 - \frac{1}{3}(\partial_x u)^2\right)\right\},$$

$$\partial_t p = -\partial_x \left\{ (h+p)u + \frac{1}{3}(\partial_x h + \partial_x p)\partial_x u + \frac{1}{3}(h+p)\partial_{xx} u \right\}$$

Yang ekuivalen dengan :

$$\partial_t u = g\partial_x p + u\partial_x u - \frac{1}{3}(\partial_x u)(\partial_{xx} u),$$

$$\partial_t p = -\partial_x u(h+p) - (\partial_x h + \partial_x p)u - \frac{1}{3}(\partial_{xx} h + \partial_{xx} p)\partial_x u - \frac{2}{3}(\partial_x h + \partial_x p)\partial_{xx} u$$

$$- \frac{1}{3}(h+p)\partial_{xxx} u$$

b. Akan ditentukan β sehingga $\delta_p H(u, p) = gp + \frac{1}{2}u^2$ dan $\delta_u H(u, p) = u + \beta p u$

$$\langle \delta_p H(u, p), \eta \rangle = \int \left(gp + \frac{1}{2}u^2 \right)\eta \, dx$$

$$= \int gp\eta + \frac{1}{2}u^2\eta \, dx$$

Untuk memperoleh $H(u, p)$, integralkan $\int gp + \frac{1}{2}u^2 \, dx$ terhadap p.

$$H(u, p) = \int\int \left(gp + \frac{1}{2}u^2 \right) dx \, dp$$

$$= \int \frac{1}{2}gp^2 + \frac{1}{2}u^2 p + j(u) \, dx, \qquad \text{dimana } j(u) \text{ adalah fungsi dari u}$$

$$\frac{\partial}{\partial \varepsilon} H\left(u+\varepsilon\eta, p\right)\Big|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} \int \left(\frac{1}{2} gp^2 + \frac{1}{2}(u+\varepsilon\eta)^2 p - j\left(u+\varepsilon\eta\right)\right) dx \Big|_{\varepsilon=0}$$

$$= \int u\eta\, p - \partial_u j\, \eta\, dx$$

$$= \int \left[up - \partial_u j\right] \eta\, dx$$

Maka $\delta_u H(u, p) = up + \partial_u j$

Karena diketahui $\delta_u H(u, p) = \beta pu + u$ maka $\beta = 1$ dan $j(u) = \frac{1}{2} u^2$

$\therefore$ Diperoleh $\beta = 1$ dengan fungsi Hamiltonian $\mathrm{H}(u, p) = \int \frac{1}{2} gp^2 + \frac{1}{2} u^2 (p+1) dx$

c. Horizontal momentum

Berdasarkan no.8b diperoleh bahwa

$$\partial_t u = gp_x + uu_x$$

$$\partial_t p = -u_x - up_x - pu_x$$

Akan ditunjukkan $\frac{d}{dt} \int u(x,t)\, p(x,t) dx = 0$

$$\frac{d}{dt} \int u(x,t)\, p(x,t) dx = \int \frac{d\left(u(x,t)\, p(x,t)\right)}{dt} dx$$

$$= \int \left(\left(\partial_t u\right) p + u \left(\partial_t p\right)\right) dx$$

$$= \int \left(\left(gp_x + uu_x\right) p + \left(-u_x - up_x - pu_x\right) u\right) dx$$

$$= \int gp_x p - uu_x - p_x u^2 dx$$

$$= \frac{1}{2} gp^2 - \frac{1}{2} u^2 - \int p_x u^2 dx$$