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The Space of Real Continuous Functions on $[a,b]$ as n-Normed Space

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Article Info

Article history:
Received April 4, 2013
Revised
Accepted

ABSTRACT

In this paper, we will show that $C[a,b]$, the space of real continuous functions on $[a,b]$, can be equipped with an n-norm $\|\cdot,\ldots,\cdot\|_\infty$ which makes $(C[a,b],\|\cdot,\ldots,\cdot\|_\infty)$ a complete n-normed space. Then we will prove a Fixed Point Theorem in $(C[a,b],\|\cdot,\ldots,\cdot\|_\infty)$. The proof does not use a Cauchy sequence, but uses a relationship between convergent sequences in norm and convergent sequence in the n-norm.

Key word: n-norm, completeness, Fixed Point Theorem

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1. INTRODUCTION

Let $n \in \mathrm{N}$ and X be a real vector space of dimension $d \geq n$. (Here d is allowed to be infinete). A real-valued function $\|\cdot,\ldots,\cdot\|$ on X^n satisfying the following four properties

(1) $\|x_1,\ldots,x_n\| = 0$ if and only if $x_1,\ldots,x_n$ are linearly dependent;
(2) $\|x_1,\ldots,x_n\|$ is invariant under permutation;
(3) $\|x_1,\ldots,x_{n-1},\alpha x_n\| = |\alpha|\|x_1,\ldots,x_{n-1},x_n\|$ for any $\alpha \in \mathrm{R}$;
(4) $\|x_1,\ldots,x_{n-1},y+z\| \leq \|x_1,\ldots,x_{n-1},y\| + \|x_1,\ldots,x_{n-1},z\|$,

is called an *n-norm* on X and the pair $(X,\|\cdot,\ldots,\cdot\|)$ is called an n-normed space.

Note that an n-normed space $(X,\|\cdot,\ldots,\cdot\|)$ satisfying the properties

(1) $\|x_1,\ldots,x_n\| \geq 0$;
(2) $\|x_1,\ldots,x_n\| = \|x_1,\ldots,x_n + \alpha_1 x_1 + \cdots + \alpha_{n-1}x_{n-1}\|$ for any $x_1,\ldots,x_n \in X$ and $\alpha_1,\ldots,\alpha_{n-1} \in \mathrm{R}$.

The concept of 2-normed space was first introduced by Gähler in the 1960's. Then 2-inner product space was introduced by Deminnie, Gähler, and White in the 1970. Afterward many researchers who study the aspects of an n-normed space, such as the topology of the n-normed space and the existence of a fixed point.

Most of the space that been studied are the space of finite dimension and standard space, for example, in [2, 3, 4], while the example for non-standard space has not been so widely studied, but one of them can be seen in [3]. This paper will discuss the space of real continuous functions on $[a, b]$, ie $C[a, b]$ as a special case of non-standard space with infinite dimensions.

This paper will show that $C[a,b]$, the space of real continuous functions on $[a,b]$, can be equipped with an n-norm $\|\cdot,\ldots,\cdot\|_\infty$ which makes $(C[a,b],\|\cdot,\ldots,\cdot\|_\infty)$ a complete n-normed space. Then it will prove a Fixed Point Theorem in $(C[a,b],\|\cdot,\ldots,\cdot\|_\infty)$. The proof does not use a Cauchy sequence, but uses a relationship between convergent sequences in norm and convergent sequence in the n-norm.

2. PRELIMINARY RESULTS

By using the properties of integral and determinant, define functions on $(C[a,b])^n$, that is:

$$\|f_1,\dots,f_n\|_p := \left(\frac{1}{n!}\int_a^b \cdots \int_a^b \left|\det\left(f_i(x_j)\right)\right|^p dx_1 \dots dx_n\right)^{\frac{1}{p}}, \text{for } 1 \le p < \infty$$

and

$$\|f_1,\dots,f_n\|_\infty := \sup_{a\le x_1\le b} \dots \sup_{a\le x_n\le b} \left|\det\left(f_i(x_j)\right)\right|, \text{for } p=\infty$$

for any $f_1,\dots,f_n \in C[a,b]$.

Fact 2.1. The inequalities

$$\|f_1,\dots,f_n\|_p \le (n!)^{1-\frac{1}{p}}\|f_1\|_p \dots \|f_n\|_p$$

and

$$\|f_1,\dots,f_n\|_\infty \le n!\ \|f_1\|_\infty \dots \|f_n\|_\infty$$

are hold for any $f_1,\dots,f_n \in C[a,b]$.

Proof. Let S_n be a set of permutation from $\{1,2,\dots,n\}$ and for any $\theta \in S_n$, define

$$sg(\theta) := \begin{cases} 1, & \text{if } \theta \text{ is even permutation} \\ -1, & \text{if } \theta \text{ is odd permutation} \end{cases}.$$

Then

$$\begin{aligned}
\|f_1,\dots,f_n\|_p &= \left(\frac{1}{n!}\int_a^b \cdots \int_a^b \left|\sum_{\theta\in S_n} sg(\theta) f_1(x_{\theta_1})\dots f_n(x_{\theta_n})\right|^p dx_{\theta_1}\dots dx_{\theta_n}\right)^{\frac{1}{p}} \\
&\le \left(\frac{1}{n!}\int_a^b \cdots \int_a^b \left(\sum_{\theta\in S_n} |f_1(x_{\theta_1})\dots f_n(x_{\theta_n})|\right)^p dx_{\theta_1}\dots dx_{\theta_n}\right)^{\frac{1}{p}} \\
&\le \left(\frac{1}{n!}\int_a^b \cdots \int_a^b \left(\sum_{\theta\in S_n} \left(\int_a^b |f_1(x_{\theta_1})\dots f_n(x_{\theta_n})|^p dx_{\theta_1}\right)^{\frac{1}{p}}\right)^p dx_{\theta_2}\dots dx_{\theta_n}\right)^{\frac{1}{p}} \\
&\ \ \vdots \\
&\le (n!)^{-\frac{1}{p}} \sum_{\theta\in S_n} \left(\int_a^b \cdots \int_a^b |f_1(x_{\theta_1})\dots f_n(x_{\theta_n})|^p dx_{\theta_1}\dots dx_{\theta_n}\right)^{\frac{1}{p}} \\
&= (n!)^{1-\frac{1}{p}} \left(\int_a^b \cdots \int_a^b |f_1(x_1)\dots f_n(x_n)|^p dx_1\dots dx_n\right)^{\frac{1}{p}} \\
&= (n!)^{1-\frac{1}{p}} \|f_1\|_p \dots \|f_n\|_p.
\end{aligned}$$

Furthermore

$$\begin{aligned}
\|f_1,\dots,f_n\|_\infty &= \sup_{a\le x_{\theta_1}\le b} \dots \sup_{a\le x_{\theta_n}\le b} \left|\sum_{\theta\in S_n} sg(\theta) f_1(x_{\theta_1})\dots f_n(x_{\theta_n})\right| \\
&\le \sup_{a\le x_{\theta_1}\le b} \dots \sup_{a\le x_{\theta_n}\le b} \left(\sum_{\theta\in S_n} |f_1(x_{\theta_1})\dots f_n(x_{\theta_n})|\right) \\
&\le \sum_{\theta\in S_n} \left(\sup_{a\le x_{\theta_1}\le b} |f_1(x_{\theta_1})| \dots \sup_{a\le x_{\theta_n}\le b} |f_n(x_{\theta_n})|\right) \\
&= \sum_{\theta\in S_n} \|f_1\|_\infty \dots \|f_n\|_\infty \\
&= n!\, \|f_1\|_\infty \dots \|f_n\|_\infty.
\end{aligned}$$

Theorem 2.2. The functions $\|\cdot,\dots,\cdot\|_p$ and $\|\cdot,\dots,\cdot\|_\infty$ define an n-norm on $C[a,b]$.

Proof. We will verify that $\|\cdot,\dots,\cdot\|_\infty$ satisfies the four properties of an n-norm.

(1) If $f_1,\dots,f_n$ are linearly dependent, then $\|f_1,\dots,f_n\|_\infty = 0$. Conversely if $\|f_1,\dots,f_n\|_\infty = 0$, then $\det\left(f_i(x_j)\right)$ for any $x_1,\dots,x_n \in [a,b]$. Hence $\{(f_1(x_1),\dots,f_1(x_n)),\dots,(f_n(x_1),\dots,f_n(x_n))\}$ is linearly dependent for any $x_1,\dots,x_n \in [a,b]$. Thus $f_1,\dots,f_n$ are linearly dependent.

(2) $\|f_1,\dots,f_n\|_\infty$ is invariant under permutation.

(3) Observe that

$$\begin{aligned}\|kf_1,\dots,f_n\|_\infty &= \sup_{a\le x_1\le b}\cdots\sup_{a\le x_n\le b}\left|k\det\left(f_i(x_j)\right)\right|\\ &= |k|\sup_{a\le x_1\le b}\cdots\sup_{a\le x_n\le b}\left|\det\left(f_i(x_j)\right)\right|\\ &= |k|\|f_1,\dots,f_n\|_\infty.\end{aligned}$$

(4) Observe that

$\|f+f',f_1,\dots,f_{n-1}\|_\infty$

$$\begin{aligned}&= \sup_{a\le x_{\theta_1}\le b}\cdots\sup_{a\le x_{\theta_n}\le b}\left|\sum_{\theta\in S_n} sg(\theta)[f(x_{\theta_1})+f'(x_{\theta_1})]f_1(x_{\theta_2})\cdots f_{n-1}(x_{\theta_n})\right|\\ &= \sup_{a\le x_{\theta_1}\le b}\cdots\sup_{a\le x_{\theta_n}\le b}\left|\sum_{\theta\in S_n} sg(\theta)f(x_{\theta_1})\cdots f_{n-1}(x_{\theta_n})+\sum_{\theta\in S_n} sg(\theta)f'^{(x_{\theta_1})}\cdots f_{n-1}(x_{\theta_n})\right|\\ &\le \sup_{a\le x_{\theta_1}\le b}\cdots\sup_{a\le x_{\theta_n}\le b}\left(\left|\sum_{\theta\in S_n} sg(\theta)f(x_{\theta_1})\cdots f_{n-1}(x_{\theta_n})\right|+\left|\sum_{\theta\in S_n} sg(\theta)f'^{(x_{\theta_1})}\cdots f_{n-1}(x_{\theta_n})\right|\right)\\ &\le \sup_{a\le x_{\theta_1}\le b}\cdots\sup_{a\le x_{\theta_n}\le b}\left|\sum_{\theta\in S_n} sg(\theta)f(x_{\theta_1})\cdots f_{n-1}(x_{\theta_n})\right|\\ &\qquad+\sup_{a\le x_{\theta_1}\le b}\cdots\sup_{a\le x_{\theta_n}\le b}\left|\sum_{\theta\in S_n} sg(\theta)f'^{(x_{\theta_1})}\cdots f_{n-1}(x_{\theta_n})\right|\\ &= \|f,f_1,\dots,f_{n-1}\|_\infty+\|f',f_1,\dots,f_{n-1}\|_\infty\end{aligned}$$

Therefore $\|\cdot,\dots,\cdot\|_\infty$ defines an n-norm on $C[a,b]$. By using Minskowski inequality, we can show that $\|\cdot,\dots,\cdot\|_p$ defines an n-norm on $C[a,b]$.

Corollary 2.3. The pair $(C[a,b], \|\cdot,\dots,\cdot\|_\infty)$ is an n-normed space.

Corollary 2.4. The pair $(C[a,b], \|\cdot,\dots,\cdot\|_p)$ is an n-normed space.

3. RESULT AND ANALYSIS

Let $\{a_1,\dots,a_n\}$ be a set of linearly independent in $C[a,b]$, we can define functions on $C[a,b]$, such as

$$\|f\|_p^{**} = \left(\sum_{\{i_2,\dots,i_n\}\subseteq\{1,\dots,n\}}\left(\left\|f,a_{i_2},\dots,a_{i_n}\right\|_p\right)^p\right)^{\frac{1}{p}}, \text{ for } 1\le p<\infty$$

and

$$\|f\|_\infty^{**} = \max_{\{i_2,\dots,i_n\}\subseteq\{1,\dots,n\}}\left\{\left\|f,a_{i_2},\dots,a_{i_n}\right\|_\infty\right\}, \text{ for } p=\infty,$$

For

$$a_i(x) = \begin{cases}\dfrac{(x-a)-\frac{(i-1)(b-a)}{n}}{\frac{b-a}{2n}}, & a+\dfrac{(i-1)(b-a)}{n}\le x<a+\dfrac{\left(i-\frac{1}{2}\right)(b-a)}{n}\\[2ex] \dfrac{(x-a)-\frac{i(b-a)}{n}}{\frac{b-a}{2n}}, & a+\dfrac{\left(i-\frac{1}{2}\right)(b-a)}{n}\le x\le a+\dfrac{i(b-a)}{n}\\[2ex] 0, & \text{the others } x.\end{cases} \tag{3.1}$$

we find a fact.

Fact 3.1. The functions $\|\cdot\|_\infty^{**}$ and $\|\cdot\|_p^{**}$ define a norm on $C[a,b]$.
Proof. Let $f, g \in C[a,b]$. If $f = 0$ then $\|f\|_\infty^{**} = 0$. Conversely, if $\|f\|_\infty^{**} = 0$, then

$$\|f, a_1, a_2, \dots, a_{n-1}\|_\infty = 0 \tag{3.2}$$
$$\|f, a_2, a_3, \dots, a_n\|_\infty = 0 \tag{3.3}$$
$$\|f, a_1, a_3, \dots, a_n\|_\infty = 0 \tag{3.4}$$

and so on until to obtain

$$\|f, a_1, \dots, a_{n-2}, a_n\|_\infty = 0 \tag{3.5}$$

By Equation (3.2) we find that f is linearly combination from $\{a_1, a_2, \dots, a_{n-1}\}$, let

$$f = \sum_{i=1}^{n-1} k_i a_i.$$

By Equation (3.3) we find that $k_1 = 0$, then by Equation (3.4) we ind that $k_2 = 0$, and so on until we obtain that $k_{n-1} = 0$. Thus $f = 0$. Furthermore, for $k \in \mathbb{R}$, we have

$$\begin{aligned}\|kf\|_\infty^{**} &= \max_{\{i_2,\dots,i_n\}\subseteq\{1,\dots,n\}} \left\{\left\|kf, a_{i_2}, \dots, a_{i_n}\right\|_\infty\right\} \\ &= |k| \max_{\{i_2,\dots,i_n\}\subseteq\{1,\dots,n\}} \left\{\left\|f, a_{i_2}, \dots, a_{i_n}\right\|_\infty\right\} \\ &= |k|\|f\|_\infty^{**}.\end{aligned}$$

Finally,

$$\begin{aligned}\|f+g\|_\infty^{**} &= \max_{\{i_2,\dots,i_n\}\subseteq\{1,\dots,n\}} \left\{\left\|f+g, a_{i_2}, \dots, a_{i_n}\right\|_\infty\right\} \\ &= \max_{\{i_2,\dots,i_n\}\subseteq\{1,\dots,n\}} \left\{\left\|f, a_{i_2}, \dots, a_{i_n}\right\|_\infty + \left\|g, a_{i_2}, \dots, a_{i_n}\right\|_\infty\right\} \\ &= \max_{\{i_2,\dots,i_n\}\subseteq\{1,\dots,n\}} \left\{\left\|f, a_{i_2}, \dots, a_{i_n}\right\|_\infty\right\} + \max_{\{i_2,\dots,i_n\}\subseteq\{1,\dots,n\}} \left\{\left\|g, a_{i_2}, \dots, a_{i_n}\right\|_\infty\right\} \\ &= \|f\|_\infty^{**} + \|g\|_\infty^{**}.\end{aligned}$$

Therefore $\|\cdot\|_\infty^{**}$ defines a norm on $C[a, b]$. By using Minskowski inequality, we can show that $\|\cdot\|_p^{**}$ defines a norm on $C[a, b]$.

Fact 3.2. Norm $\|\cdot\|_\infty^{**}$ is equivalent to norm $\|\cdot\|_\infty$ on $C[a, b]$.
Proof. Note that $\|a_1\|_\infty = \dots = \|a_n\|_\infty = 1$. Let $f \in C[a,b]$ and $\{i_2, \dots, i_n\} \subseteq \{1, \dots, n\}$, then

$$\left\|f, a_{i_2}, \dots, a_{i_n}\right\|_\infty \le n!\,\|f\|_\infty \left\|a_{i_2}\right\|_\infty \cdots \left\|a_{i_n}\right\|_\infty = n!\,\|f\|_\infty.$$

Thus

$$\|f\|_\infty^{**} = \max_{\{i_2,\dots,i_n\}\subseteq\{1,\dots,n\}} \left\{\left\|f, a_{i_2}, \dots, a_{i_n}\right\|_\infty\right\} \le \max_{\{i_2,\dots,i_n\}\subseteq\{1,\dots,n\}} \{n!\,\|f\|_\infty, \dots, n!\,\|f\|_\infty\} = n!\,\|f\|_\infty.$$

Furthermore

$$\begin{aligned}\|f, a_1, \dots, a_{n-1}\|_\infty &= \sup_{a\le x_1\le b} \cdots \sup_{a\le x_n\le b} \left|\det\begin{bmatrix} f(x_1) & f(x_2) & \cdots & f(x_n) \\ a_1(x_1) & a_1(x_2) & \cdots & a_1(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1}(x_1) & a_{n-1}(x_2) & \cdots & a_{n-1}(x_n)\end{bmatrix}\right| \\ &\ge \sup_{a+\frac{(b-a)}{n}\le x_1\le b} \cdots \sup_{a+\frac{(n-1)(b-a)}{n}\le x_n\le b} \left|\det\begin{bmatrix} f(x_1) & f(x_2) & \cdots & f(x_n) \\ a_1(x_1) & a_1(x_2) & \cdots & a_1(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1}(x_1) & a_{n-1}(x_2) & \cdots & a_{n-1}(x_n)\end{bmatrix}\right| \\ &= \sup_{a+\frac{(b-a)}{n}\le x_1\le b} \cdots \sup_{a+\frac{(n-1)(b-a)}{n}\le x_n\le b} |f(x_1)a_1(x_2)\dots a_{n-1}(x_n)| \\ &= \sup_{a+\frac{(b-a)}{n}\le x_1\le b} |f(x_1)|.\end{aligned}$$

By similar method we find that

$$\|f, a_2, \dots, a_n\|_\infty \ge \sup_{a+\frac{2(b-a)}{n}\le x_2\le b} |f(x_2)|$$

$$\|f, a_1, a_3, \dots, a_n\|_\infty \ge \sup_{a+\frac{3(b-a)}{n}\le x_3\le b} |f(x_3)|$$

and so on until we obtain that

$$\|f, a_1, \dots, a_{n-2}, a_n\|_\infty \ge \sup_{a\le x_n\le a+\frac{(n-1)(b-a)}{n}} |f(x_n)|.$$

Thus

$$\begin{aligned}\|f\|_\infty^{**} &= \max_{\{i_2,\dots,i_n\}\subseteq\{1,\dots,n\}}\left\{\|f,a_{i_2},\dots,a_{i_n}\|_\infty\right\}\\ &\geq \max\left\{\sup_{a+\frac{(b-a)}{n}\leq x_1\leq b}|f(x_1)|,\dots,\sup_{a\leq x_n\leq a+\frac{(n-1)(b-a)}{n}}|f(x_n)|\right\}\\ &= \max\left\{\sup_{a+\frac{(b-a)}{n}\leq x\leq b}|f(x)|,\dots,\sup_{a\leq x\leq a+\frac{(n-1)(b-a)}{n}}|f(x)|\right\}\\ &= \sup_{a\leq x\leq b}|f(x)|\\ &= \|f\|_\infty.\end{aligned}$$

Therefore we have $\|f\|_\infty \leq \|f\|_\infty^{**} \leq n!\,\|f\|_\infty$, so that norm $\|\cdot\|_\infty^{**}$ is equivalent to norm $\|\cdot\|_\infty$ on $C[a, b]$.

The main result from this paper are The Completeness of $C[a, b]$ and A Fixed Point Theorem on $C[a, b]$. First, we will prove a lemma that will help in proving The Completeness of $C[a, b]$ and A Fixed Point Theorem on $C[a, b]$.

Lemma 3.3. A sequence in $C[a, b]$ is convergent in the n-norm $\|\cdot,\dots,\cdot\|_\infty$ if and only if it is convergent in the norm $\|\cdot\|_\infty$. Moreover a sequence in $C[a, b]$ is Cauchy in the n-norm $\|\cdot,\dots,\cdot\|_\infty$ if and only if it is Cauchy in the norm $\|\cdot\|_\infty$.

Proof. Let (f_n) is convergent sequence in $C[a, b]$ to $f \in C[a,b]$ in the norm $\|\cdot\|_\infty$, then

$$\lim_{n\to\infty}\|f_n - f\|_\infty = 0,$$

so that for $g_1 g_2, \dots, g_{n-1} \in C[a,b]$ satisfies

$$\lim_{n\to\infty}\|f_n - f, g_1 g_2, \dots, g_{n-1}\|_\infty \leq \lim_{n\to\infty}(n!\,\|f_n - f\|_\infty \|g_1\|_\infty \cdots \|g_{n-1}\|_\infty) = 0.$$

Therefore (f_n) is convergent in $\|\cdot,\dots,\cdot\|_\infty$. Conversely let (f_n) is convergent sequence in $C[a, b]$ to $f \in C[a,b]$ in the n-norm $\|\cdot,\dots,\cdot\|_\infty$, then

$$\lim_{n\to\infty}\|f_n - f, g_1 g_2, \dots, g_{n-1}\|_\infty = 0 \text{ for any } g_1 g_2, \dots, g_{n-1} \in C[a,b],$$

so that for any $\{i_2, \dots, i_n\} \subseteq \{1, \dots, n\}$ and $\{a_1, \dots, a_n\}$ in the Equation (3.1) satisfy

$$\lim_{n\to\infty}\left\|f_n - f, a_{i_1} a_{i_2}, \dots, a_{i_{n-1}}\right\|_\infty = 0.$$

Then

$$\lim_{n\to\infty}\|f_n - f\|_\infty \leq \lim_{n\to\infty}\|f_n - f\|_\infty^{**} = 0.$$

Therefore (f_n) is convergent in the norm $\|\cdot\|_\infty$.

Theorem 3.4. $(C[a, b], \|\cdot,\dots,\cdot\|_\infty)$ is a complete space.

Proof. Let (f_n) is a Cauchy sequence in $C[a, b]$ in the n-norm $\|\cdot,\dots,\cdot\|_\infty$, by Lemma 3.3. (f_n) is a Cauchy sequence in the norm $\|\cdot\|_\infty$. Because $(C[a, b], \|\cdot\|_\infty)$ is a complete space, (f_n) is convergent to a function $f \in C[a,b]$ in the norm $\|\cdot\|_\infty$, so that (f_n) is convergent to f in the n-norm $\|\cdot,\dots,\cdot\|_\infty$. Therefore $(C[a, b], \|\cdot,\dots,\cdot\|_\infty)$ is a complete space.

Consequenly, we have the following result.

Theorem 3.5. (Fixed Point Theorem) Let $\{a_1, \dots, a_n\}$ is a set of linearly independent in $(C[a, b], \|\cdot,\dots,\cdot\|_\infty)$ in the Equation (3.1) and T is a contractive mapping of $C[a, b]$ in to itself, that is, there exists a constant $k \in (0, 1)$ such that

$$\left\|Tf - Tg, a_{i_1} a_{i_2}, \dots, a_{i_{n-1}}\right\|_\infty \leq k\left\|-g, a_{i_1} a_{i_2}, \dots, a_{i_{n-1}}\right\|_\infty$$

for any $f, g \in C[a,b]$ and $\{i_2, \dots, i_n\} \subseteq \{1, \dots, n\}$. Then T has a unique fixed point in $C[a,b]$.

We will show that $(C[a, b], \|\cdot\|_\infty^{**})$ is Banach space before we prove Theorem 3.5.

Let (f_n) is a Cauchy sequence in $C[a, b]$ in the norm $\|\cdot\|_\infty^{**}$. Then (f_n) is a Cauchy sequence in $C[a, b]$ in the norm $\|\cdot\|_\infty$ because norm $\|\cdot\|_\infty^{**}$ is equivalent to norm $\|\cdot\|_\infty$ on $C[a, b]$, so that (f_n) is convergent to a function $f \in C[a,b]$ in the norm $\|\cdot\|_\infty$. Then (f_n) is convergent to f in $C[a,b]$ in the norm $\|\cdot\|_\infty^{**}$ because norm $\|\cdot\|_\infty^{**}$ is equivalent to norm $\|\cdot\|_\infty$ on $C[a, b]$. Thus $(C[a, b], \|\cdot\|_\infty^{**})$ is Banach space.

Proof. Consider that

$$\begin{aligned}\|Tf - Tg\|_\infty^{**} &= \max_{\{i_2,\dots,i_n\}\subseteq\{1,\dots,n\}}\left\{\|Tf - Tg, a_{i_2}, \dots, a_{i_n}\|_\infty\right\}\\ &\leq \max_{\{i_2,\dots,i_n\}\subseteq\{1,\dots,n\}}\left\{k\|f - g, a_{i_2}, \dots, a_{i_n}\|_\infty\right\}\\ &\leq k\max_{\{i_2,\dots,i_n\}\subseteq\{1,\dots,n\}}\left\{\|f - g, a_{i_2}, \dots, a_{i_n}\|_\infty\right\}\\ &= k\|f - g\|_\infty^{**}.\end{aligned}$$

Thus T is contractive mapping in the norm $\|\cdot\|_\infty^{**}$, so that by Fixed Point Theorem on Banach space T has a unique fixed point in $C[a,b]$.

4. CONCLUSION

According to Gähler, in every normed space can be defined an n-norm. This paper has shown that $C[a, b]$ can be equipped with an n-norm $\|\cdot, \ldots, \cdot\|_p$. Then, inspired by it, defined n-norm $\|\cdot, \ldots, \cdot\|_\infty$, so that the pair $(C[a, b], \|\cdot, \ldots, \cdot\|_\infty)$ is an n-normed space. Furtherrmore $(C[a, b], \|\cdot, \ldots, \cdot\|_\infty)$ is a complete space. It is derived from the fact that the sequence in $C[a, b]$ converges in the n-norm $\|\cdot, \ldots, \cdot\|_\infty$ if and only if the sequence is also convergent in the norm $\|\cdot\|_\infty$. Moreover the sequence in $C[a, b]$ is Cauchy in the the n-norm $\|\cdot, \ldots, \cdot\|_\infty$ if and only if the sequence is also Cauchy in the norm $\|\cdot\|_\infty$.

In addition it has been proved a Fixed Point Theorem in $(C[a, b], \|\cdot, \ldots, \cdot\|_\infty)$. The method of proof of Fixed Point Theorem in $(C[a, b], \|\cdot, \ldots, \cdot\|_\infty)$ uses a relationship between convergent sequences in norm $\|\cdot\|_\infty$ and norm $\|\cdot\|_\infty^{**}$ that be induced from n-norm $\|\cdot, \ldots, \cdot\|_\infty$.

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