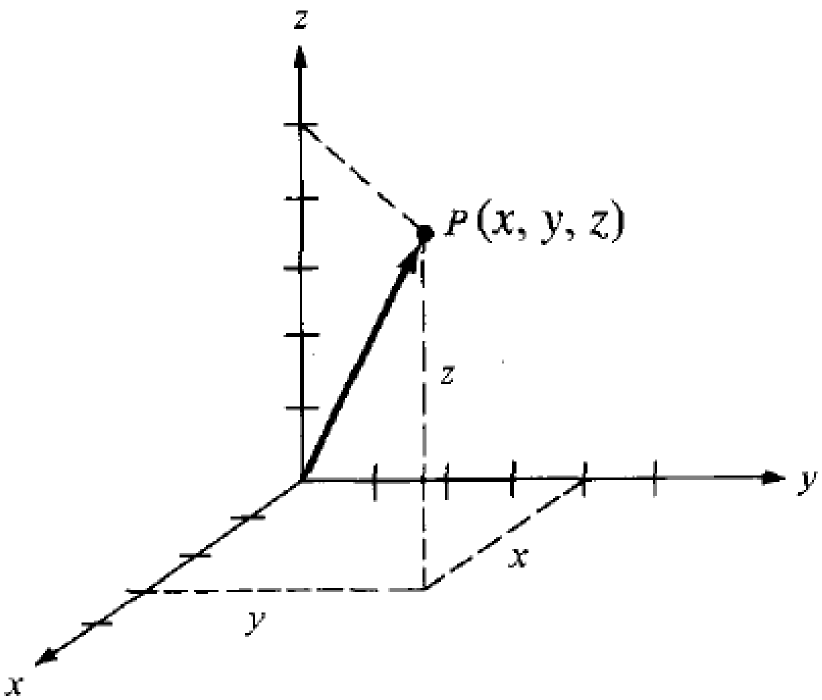
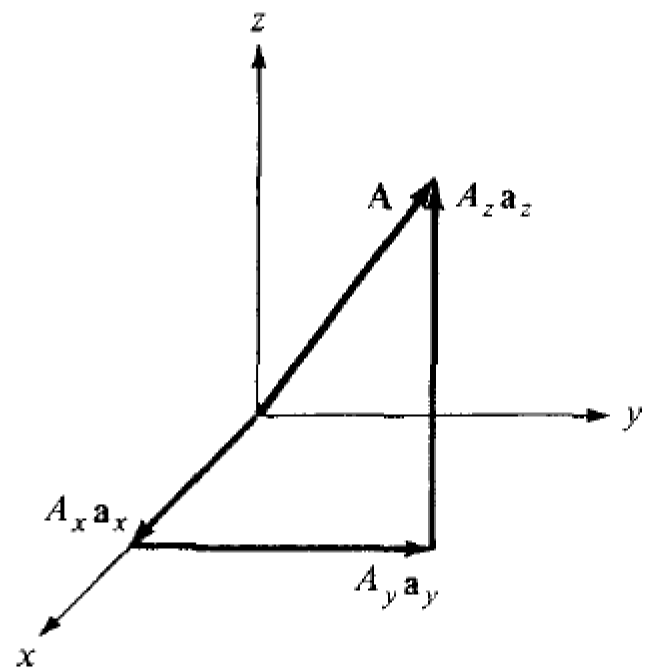
2. SISTEM KOORDINAT DAN TRANSFORMASI

2.1. KOORDINAT CARTESIAN (X,Y,Z)





$$-\infty < x < \infty$$

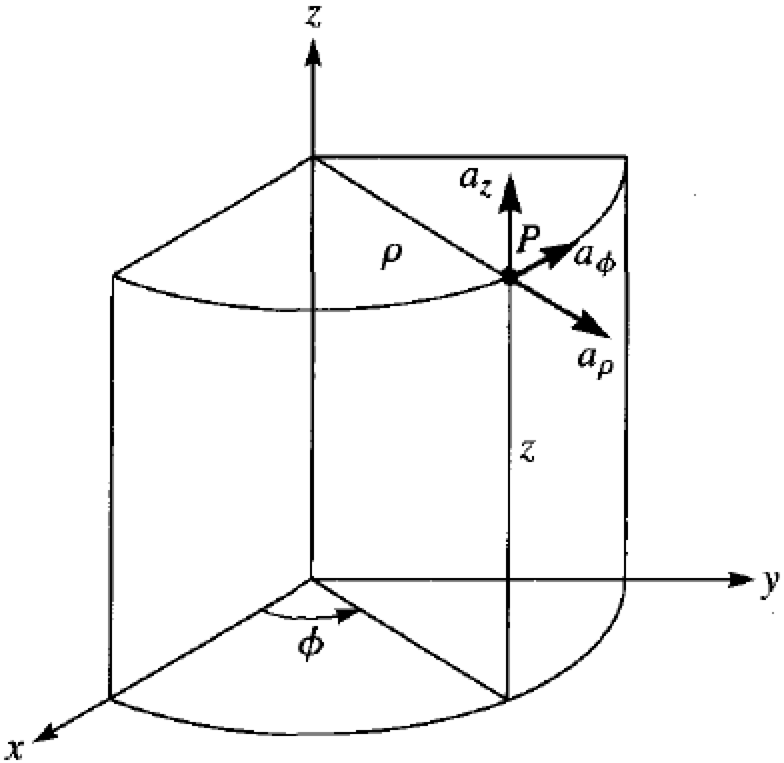
$$-\infty < y < \infty$$

$$-\infty < z < \infty$$

$$\mathbf{A} = (A_x, A_y, A_z)$$

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

2.2. KOORDINAT SILINDER (ρ, ϕ, Z)



$$0 \leq \rho < \infty$$

$$0 \leq \phi < 2\pi$$

$$-\infty < z < \infty$$

ρ = jari-jari silinder melalui P atau jarak radial dari sumbu-z

ϕ = sudut azimut, diukur dari sumbu-x pada bidang xy

z = sama dengan pada koordinat Cartesian

$$\mathbf{A} = (A_\rho, A_\phi, A_z)$$

$$\mathbf{A} = A_\rho \mathbf{a}_\rho + A_\phi \mathbf{a}_\phi + A_z \mathbf{a}_z$$

$$|\mathbf{A}| = (A_\rho^2 + A_\phi^2 + A_z^2)^{1/2}$$

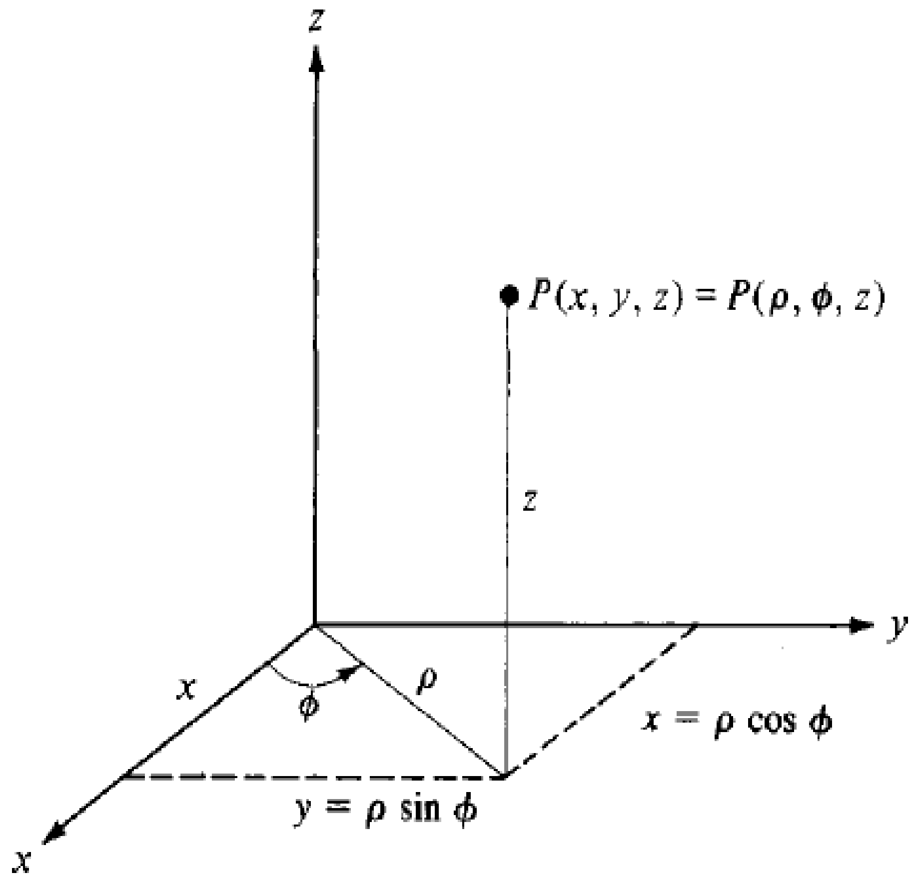
$$\mathbf{a}_\rho \cdot \mathbf{a}_\rho = \mathbf{a}_\phi \cdot \mathbf{a}_\phi = \mathbf{a}_z \cdot \mathbf{a}_z = 1$$

$$\mathbf{a}_\rho \cdot \mathbf{a}_\phi = \mathbf{a}_\phi \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_\rho = 0$$

$$\mathbf{a}_\rho \times \mathbf{a}_\phi = \mathbf{a}_z$$

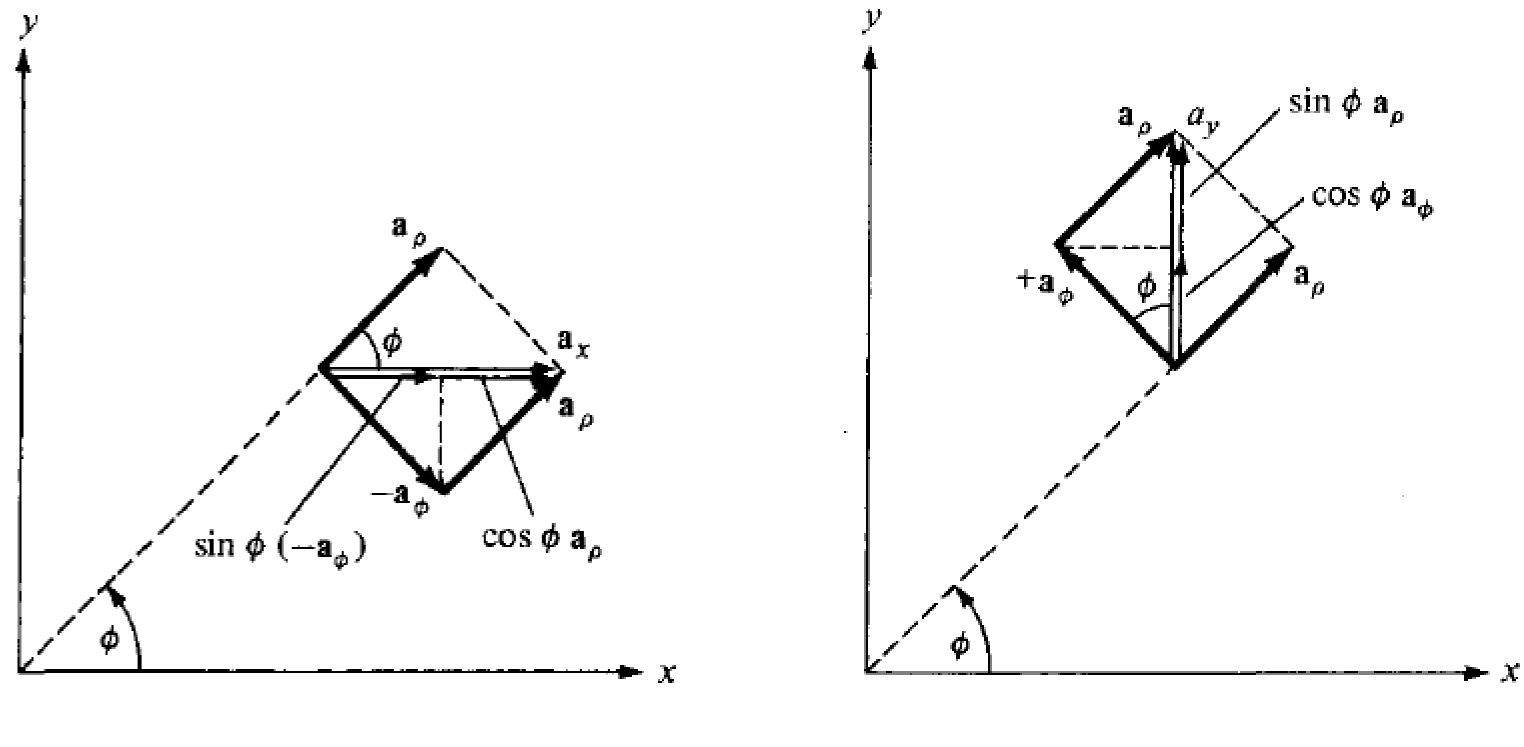
$$\mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_\rho$$

$$\mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi$$



$$\rho = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}, \qquad z = z$$

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z$$



(a)

(b)

$$\mathbf{a}_x = \cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_y = \sin\phi\,\mathbf{a}_\rho + \cos\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_z = \mathbf{a}_z$$

$$\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$$

$$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$$

$$\mathbf{a}_z = \mathbf{a}_z$$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$A_\rho = A_x \cos\phi + A_y \sin\phi$$

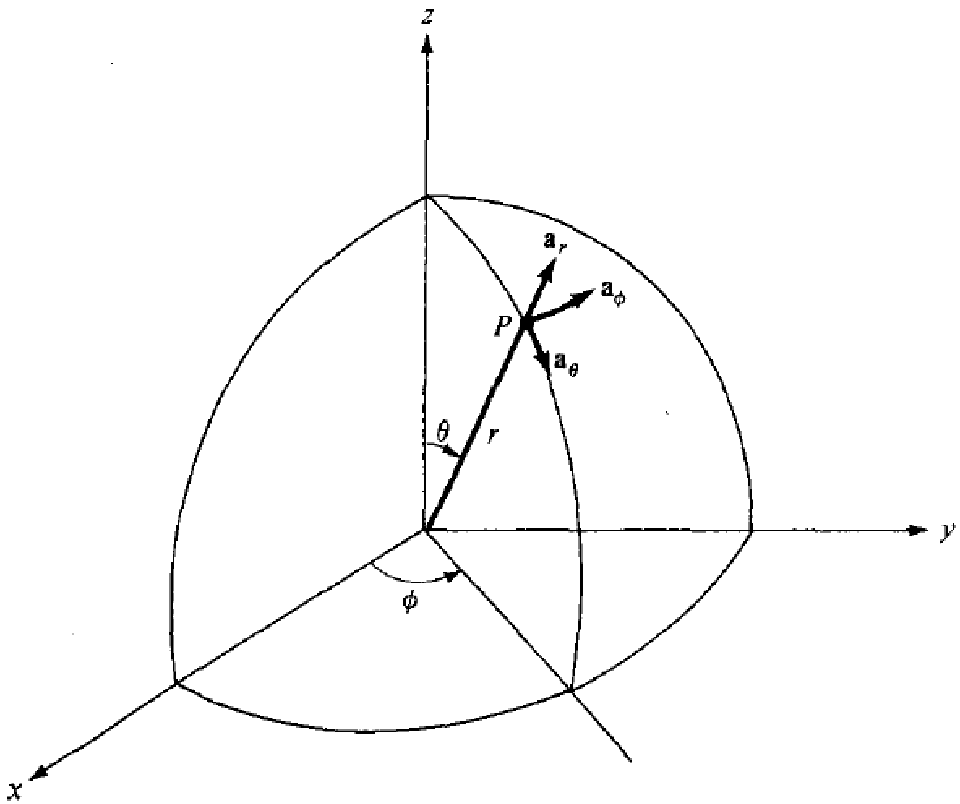
$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

$$A_z = A_z$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\mathbf{A} = (A_x \cos\phi + A_y \sin\phi)\mathbf{a}_\rho + (-A_x \sin\phi + A_y \cos\phi)\mathbf{a}_\phi + A_z\mathbf{a}_z$$

2.3. KOORDINAT BOLA (r, θ, ϕ)



r = jari-jari bola dengan pusat di titik asal dan melalui P;

θ = (disebut *colatitude*), sudut antara sumbu-z dan vektor posisi P;

ϕ = diukur dari sumbu-x (sama seperti koordinat silinder)

$$\mathbf{A} = (A_r, A_\theta, A_\phi)$$

$$\mathbf{A} = A_r\mathbf{a}_r + A_\theta\mathbf{a}_\theta + A_\phi\mathbf{a}_\phi$$

$$|\mathbf{A}| = (A_r^2 + A_\theta^2 + A_\theta^2)^{1/2}$$

$$0 \le r < \infty$$

$$0 \le \theta \le \pi$$

$$0 \le \phi < 2\pi$$

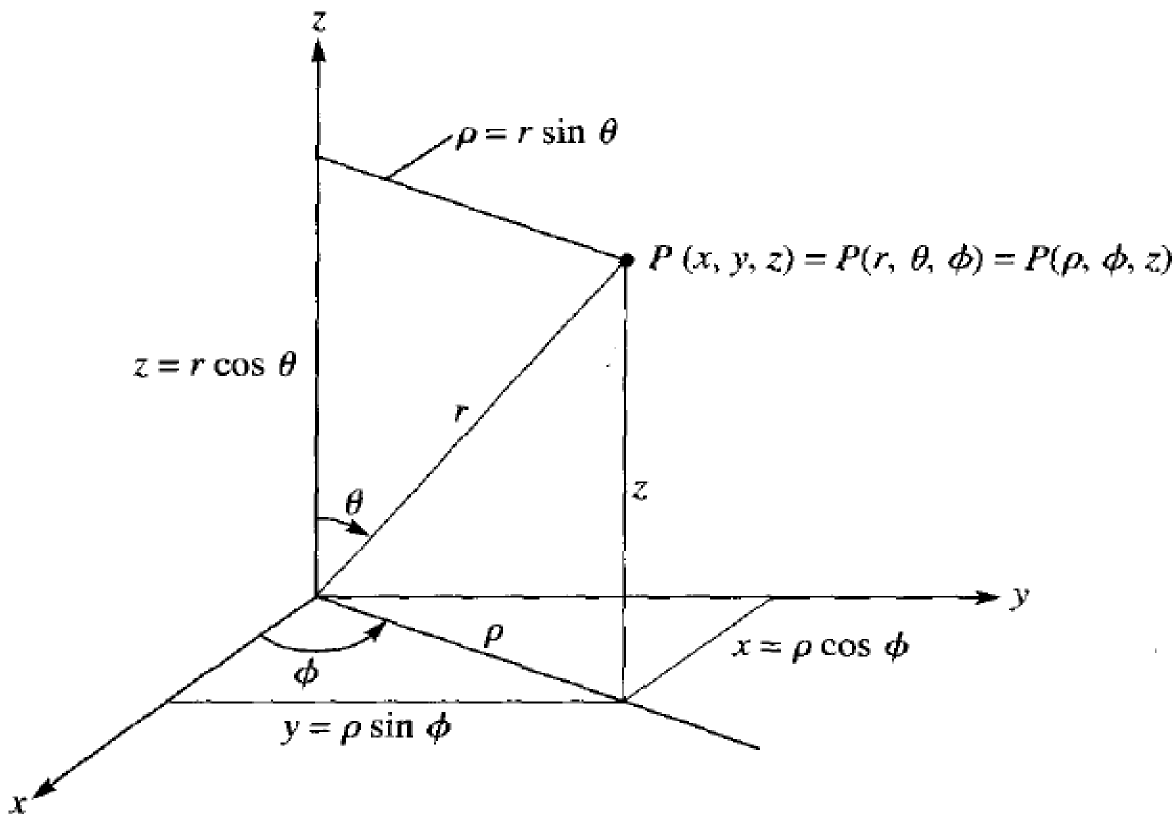
$$\mathbf{a}_r \cdot \mathbf{a}_r = \mathbf{a}_\theta \cdot \mathbf{a}_\theta = \mathbf{a}_\phi \cdot \mathbf{a}_\phi = 1$$

$$\mathbf{a}_r \cdot \mathbf{a}_\theta = \mathbf{a}_\theta \cdot \mathbf{a}_\phi = \mathbf{a}_\phi \cdot \mathbf{a}_r = 0$$

$$\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$$

$$\mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_r$$

$$\mathbf{a}_\phi \times \mathbf{a}_r = \mathbf{a}_\theta$$



$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

$$\mathbf{a}_x = \sin\theta\cos\phi\,\mathbf{a}_r + \cos\theta\cos\phi\,\mathbf{a}_\theta - \sin\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_y = \sin\theta\sin\phi\,\mathbf{a}_r + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_z = \cos\theta\,\mathbf{a}_r - \sin\theta\,\mathbf{a}_\theta$$

$$\mathbf{a}_r = \sin\theta\cos\phi\,\mathbf{a}_x + \sin\theta\sin\phi\,\mathbf{a}_y + \cos\theta\,\mathbf{a}_z$$

$$\mathbf{a}_\theta = \cos\theta\cos\phi\,\mathbf{a}_x + \cos\theta\sin\phi\,\mathbf{a}_y - \sin\theta\,\mathbf{a}_z$$

$$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

EXAMPLE

Given point $P(-2, 6, 3)$ and vector $\mathbf{A} = y\mathbf{a}_x + (x + z)\mathbf{a}_y$, express P and $\mathbf{A}$ in cylindrical and spherical coordinates. Evaluate $\mathbf{A}$ at P in the Cartesian, cylindrical, and spherical systems.

Solution:

At point P: $x = -2, y = 6, z = 3$. Hence,

$$\rho = \sqrt{x^2 + y^2} = \sqrt{4 + 36} = 6.32$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{6}{-2} = 108.43^\circ$$

$$z = 3$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4 + 36 + 9} = 7$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1}\frac{\sqrt{40}}{3} = 64.62^\circ$$

Thus,

$$P(-2, 6, 3) = P(6.32, 108.43^\circ, 3) = P(7, 64.62^\circ, 108.43^\circ)$$

In the Cartesian system, **A** at P is

$$\mathbf{A} = 6\mathbf{a}_x + \mathbf{a}_y$$

For vector **A**, $A_x = y$, $A_y = x + z$, $A_z = 0$. Hence, in the cylindrical system

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ x + z \\ 0 \end{bmatrix}$$

$$A_\rho = y\cos\phi + (x + z)\sin\phi$$

$$A_\phi = -y\sin\phi + (x + z)\cos\phi$$

$$A_z = 0$$

But $x = \rho\cos\phi$, $y = \rho\sin\phi$, and substituting these yields

$$\begin{aligned} \mathbf{A} = (A_\rho, A_\phi, A_z) = {} & [\rho\cos\phi\sin\phi + (\rho\cos\phi + z)\sin\phi]\mathbf{a}_\rho \\ & + [-\rho\sin^2\phi + (\rho\cos\phi + z)\cos\phi]\mathbf{a}_\phi \end{aligned}$$

At P $\quad \rho = \sqrt{40}, \qquad \tan\phi = \dfrac{6}{-2}$

Hence, $\quad \cos\phi = \dfrac{-2}{\sqrt{40}}, \qquad \sin\phi = \dfrac{6}{\sqrt{40}}$

$$\mathbf{A} = \left[\sqrt{40}\cdot\frac{-2}{\sqrt{40}}\cdot\frac{6}{\sqrt{40}} + \left(\sqrt{40}\cdot\frac{-2}{\sqrt{40}} + 3\right)\cdot\frac{6}{\sqrt{40}}\right]\mathbf{a}_\rho$$
$$+ \left[-\sqrt{40}\cdot\frac{36}{40} + \left(\sqrt{40}\cdot\frac{-2}{\sqrt{40}} + 3\right)\cdot\frac{-2}{\sqrt{40}}\right]\mathbf{a}_\phi$$
$$= \frac{-6}{\sqrt{40}}\mathbf{a}_\rho - \frac{38}{\sqrt{40}}\mathbf{a}_\phi = -0.9487\mathbf{a}_\rho - 6.008\mathbf{a}_\phi$$

Similarly, in the spherical system

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} y \\ x + z \\ 0 \end{bmatrix}$$

$$A_r = y\sin\theta\cos\phi + (x + z)\sin\theta\sin\phi$$

$$A_\theta = y\cos\theta\cos\phi + (x + z)\cos\theta\sin\phi$$

$$A_\phi = -y\sin\phi + (x + z)\cos\phi$$

But $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, and $z = r\cos\theta$. Substituting these yields

$$\begin{aligned}\mathbf{A} &= (A_r, A_\theta, A_\phi)\\ &= r[\sin^2\theta \cos\phi \sin\phi + (\sin\theta\cos\phi + \cos\theta)\sin\theta\sin\phi]\mathbf{a}_r\\ &\quad + r[\sin\theta\cos\theta\sin\phi\cos\phi + (\sin\theta\cos\phi + \cos\theta)\cos\theta\sin\phi]\mathbf{a}_\theta\\ &\quad + r[-\sin\theta\sin^2\phi + (\sin\theta\cos\phi + \cos\theta)\cos\phi]\mathbf{a}_\phi\end{aligned}$$

At P $\quad r = 7, \qquad \tan\phi = \dfrac{6}{-2}, \qquad \tan\theta = \dfrac{\sqrt{40}}{3}$

Hence,

$$\cos\phi = \frac{-2}{\sqrt{40}}, \qquad \sin\phi = \frac{6}{\sqrt{40}}, \qquad \cos\theta = \frac{3}{7}, \qquad \sin\theta = \frac{\sqrt{40}}{7}$$

$$\begin{aligned}\mathbf{A} &= 7\cdot\left[\frac{40}{49}\cdot\frac{-2}{\sqrt{40}}\cdot\frac{6}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{\sqrt{40}}{7}\cdot\frac{6}{\sqrt{40}}\right]\mathbf{a}_r\\ &\quad + 7\cdot\left[\frac{\sqrt{40}}{7}\cdot\frac{3}{7}\cdot\frac{6}{\sqrt{40}}\cdot\frac{-2}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{3}{7}\cdot\frac{6}{\sqrt{40}}\right]\mathbf{a}_\theta\\ &\quad + 7\cdot\left[\frac{-\sqrt{40}}{7}\cdot\frac{36}{40} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{-2}{\sqrt{40}}\right]\mathbf{a}_\phi\end{aligned}$$

$$\mathbf{A} = \frac{-6}{7}\mathbf{a}_r - \frac{18}{7\sqrt{40}}\mathbf{a}_\theta - \frac{38}{\sqrt{40}}\mathbf{a}_\phi$$
$$= -0.8571\mathbf{a}_r - 0.4066\mathbf{a}_\theta - 6.008\mathbf{a}_\phi$$

Note that $|\mathbf{A}|$ is the same in the three systems; that is,

$$|\mathbf{A}(x, y, z)| = |\mathbf{A}(\rho, \phi, z)| = |\mathbf{A}(r, \theta, \phi)| = 6.083$$

PRACTICE EXERCISE

(a) Convert points $P(1, 3, 5)$, $T(0, -4, 3)$, and $S(-3, -4, -10)$ from Cartesian to cylindrical and spherical coordinates.

(b) Transform vector

$$\mathbf{Q} = \frac{\sqrt{x^2 + y^2}\mathbf{a}_x}{\sqrt{x^2 + y^2 + z^2}} - \frac{yz\,\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}$$

to cylindrical and spherical coordinates.

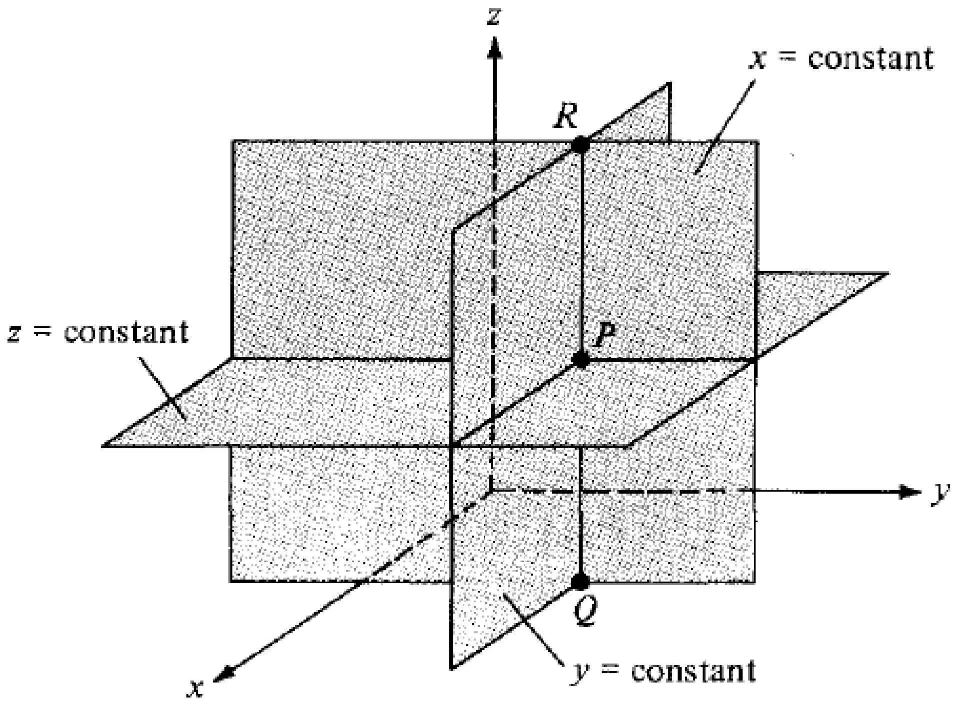
(c) Evaluate $\mathbf{Q}$ at T in the three coordinate systems.

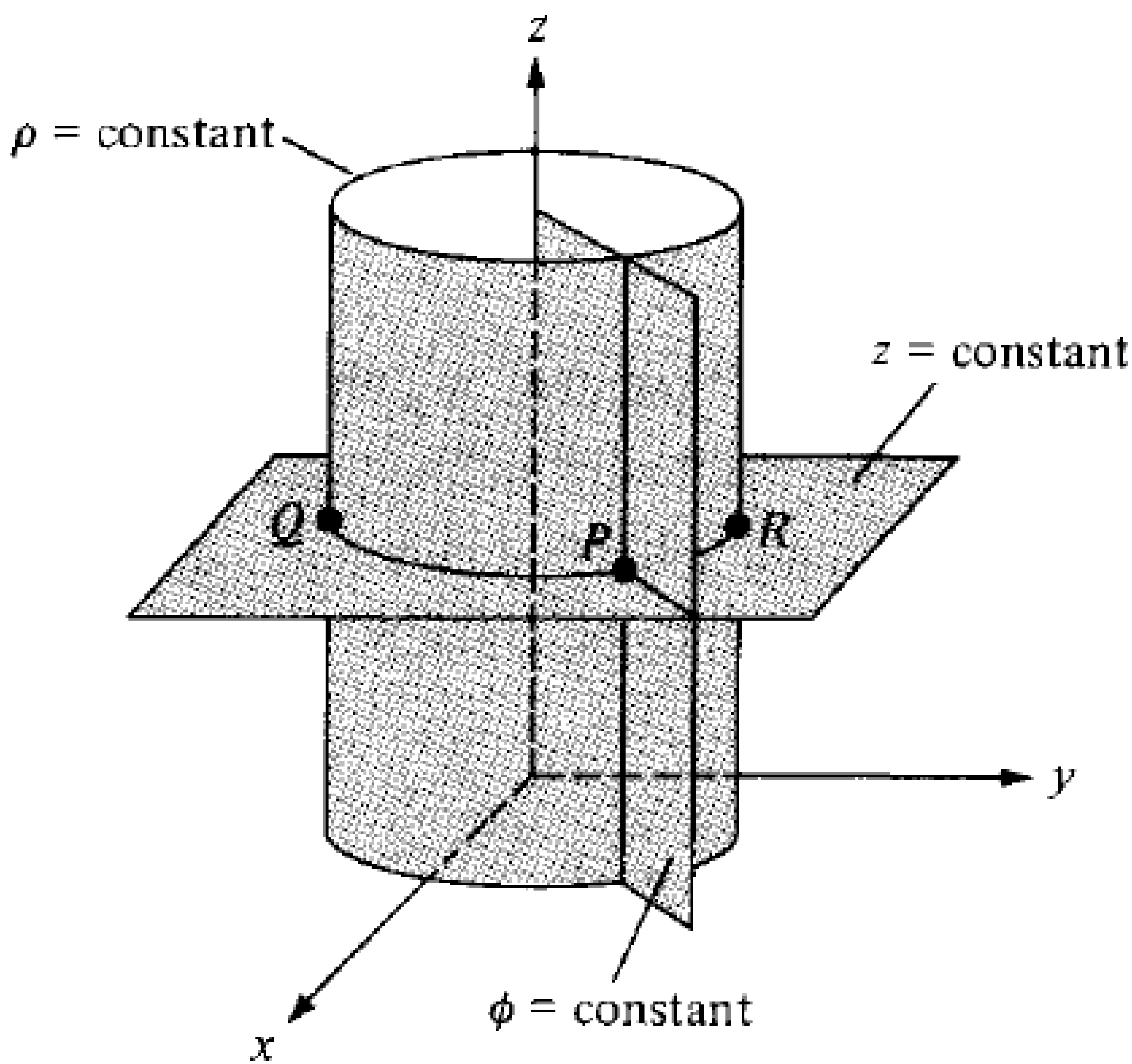
Answer: (a) $P(3.162, 71.56°, 5)$, $P(5.916, 32.31°, 71.56°)$, $T(4, 270°, 3)$, $T(5, 53.13°, 270°)$, $S(5, 233.1°, -10)$, $S(11.18, 153.43°, 233.1°)$

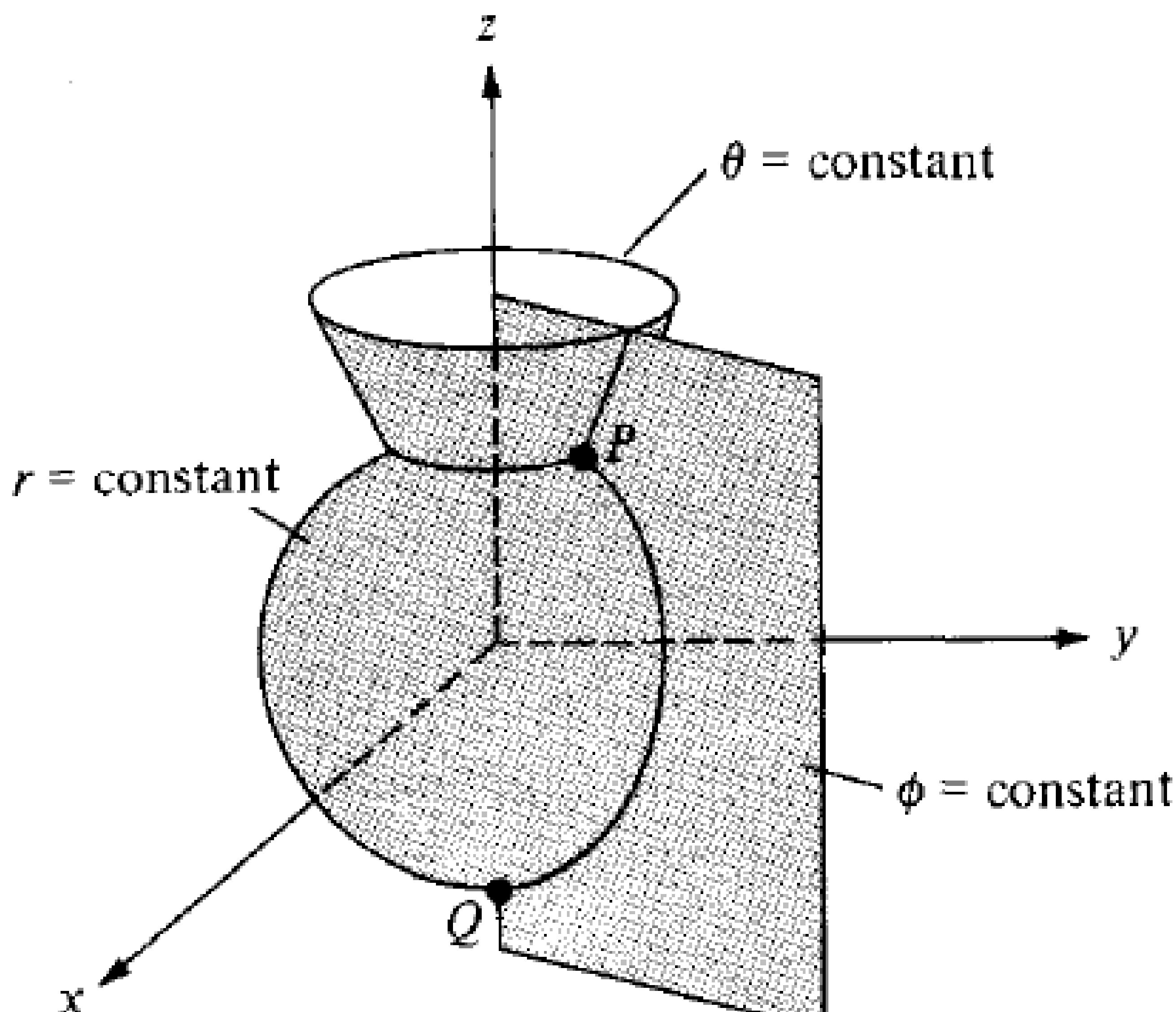
(b) $\dfrac{\rho}{\sqrt{\rho^2 + z^2}}(\cos\phi\, \mathbf{a}_\rho - \sin\phi\, \mathbf{a}_\phi - z\sin\phi\, \mathbf{a}_z)$, $\sin\theta\,(\sin\theta\cos\phi - r\cos^2\theta\sin\phi)\mathbf{a}_r + \sin\theta\cos\theta\,(\cos\phi + r\sin\theta\sin\phi)\mathbf{a}_\theta - \sin\theta\sin\phi\,\mathbf{a}_\phi$

(c) $0.8\mathbf{a}_x + 2.4\mathbf{a}_z$, $0.8\mathbf{a}_\phi + 2.4\mathbf{a}_z$, $1.44\mathbf{a}_r - 1.92\mathbf{a}_\theta + 0.8\mathbf{a}_\phi$.

2.4. PERMUKAAN KOORDINAT KONSTAN




z
ρ = constant
z = constant
Q
P
R
y
x
φ = constant


z
θ = constant
P
r = constant
y
φ = constant
Q
x

EXAMPLE

Given a vector field

$$\mathbf{D} = r \sin \phi \, \mathbf{a}_r - \frac{1}{r} \sin \theta \cos \phi \, \mathbf{a}_\theta + r^2 \mathbf{a}_\phi$$

determine

(a) $\mathbf{D}$ at $P(10, 150°, 330°)$

(b) The component of $\mathbf{D}$ tangential to the spherical surface $r = 10$ at P

(c) A unit vector at P perpendicular to $\mathbf{D}$ and tangential to the cone $\theta = 150°$

Solution:

(a) At P, $r = 10$, $\theta = 150°$, and $\phi = 330°$. Hence,

$$\mathbf{D} = 10 \sin 330° \, \mathbf{a}_r - \frac{1}{10} \sin 150° \cos 330° \, \mathbf{a}_\theta + 100 \, \mathbf{a}_\phi = (-5, 0.043, 100)$$

(b) Any vector **D** can always be resolved into two orthogonal components:

$$\mathbf{D} = \mathbf{D}_t + \mathbf{D}_n$$

where D_t is tangential to a given surface and $\mathbf{D}_n$ is normal to it. In our case, since $\mathbf{a}_r$ is normal to the surface $r = 10$,

$$\mathbf{D}_n = r \sin \phi \, \mathbf{a}_r = -5\mathbf{a}_r$$

Hence,

$$\mathbf{D}_t = \mathbf{D} - \mathbf{D}_n = 0.043\mathbf{a}_\theta + 100\mathbf{a}_\phi$$

(c) A vector at P perpendicular to **D** and tangential to the cone $\theta = 150°$ is the same as the vector perpendicular to both **D** and $\mathbf{a}_\theta$. Hence,

$$\mathbf{D} \times \mathbf{a}_\theta = \begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\theta & \mathbf{a}_\phi \\ -5 & 0.043 & 100 \\ 0 & 1 & 0 \end{vmatrix}$$
$$= -100\mathbf{a}_r - 5\mathbf{a}_\phi$$

A unit vector along this is

$$\mathbf{a} = \frac{-100\mathbf{a}_r - 5\mathbf{a}_\phi}{\sqrt{100^2 + 5^2}} = -0.9988\mathbf{a}_r - 0.0499\mathbf{a}_\phi$$

REVIEW QUESTIONS

2.1 The ranges of θ and ϕ as given by eq. (2.17) are not the only possible ones. The following are all alternative ranges of θ and ϕ, except

(a) $0 \le \theta < 2\pi, 0 \le \phi \le \pi$
(b) $0 \le \theta < 2\pi, 0 \le \phi < 2\pi$
(c) $-\pi \le \theta \le \pi, 0 \le \phi \le \pi$
(d) $-\pi/2 \le \theta \le \pi/2, 0 \le \phi < 2\pi$
(e) $0 \le \theta \le \pi, \ -\pi \le \phi < \pi$
(f) $-\pi \le \theta < \pi, \ -\pi \le \phi < \pi$

$$0 \le r < \infty$$
$$0 \le \theta \le \pi \qquad (2.17)$$
$$0 \le \phi < 2\pi$$

2.2 At Cartesian point $(-3, 4, -1)$, which of these is incorrect?

(a) $\rho = -5$
(b) $r = \sqrt{26}$
(c) $\theta = \tan^{-1} \dfrac{5}{-1}$
(d) $\phi = \tan^{-1} \dfrac{4}{-3}$

2.3 Which of these is not valid at point (0, 4, 0)?

(a) $\mathbf{a}_\phi = -\mathbf{a}_x$
(b) $\mathbf{a}_\theta = -\mathbf{a}_z$
(c) $\mathbf{a}_r = 4\mathbf{a}_y$
(d) $\mathbf{a}_\rho = \mathbf{a}_y$

2.4 A unit normal vector to the cone $\theta = 30°$ is:

(a) $\mathbf{a}_r$
(b) $\mathbf{a}_\theta$
(c) $\mathbf{a}_\phi$
(d) none of the above

2.5 At every point in space, $\mathbf{a}_\phi \cdot \mathbf{a}_\theta = 1$.

(a) True
(b) False

2.6 If $\mathbf{H} = 4\mathbf{a}_\rho - 3\mathbf{a}_\phi + 5\mathbf{a}_z$, at $(1, \pi/2, 0)$ the component of $\mathbf{H}$ parallel to surface $\rho = 1$ is

(a) $4\mathbf{a}_\rho$
(b) $5\mathbf{a}_z$
(c) $-3\mathbf{a}_\phi$
(d) $-3\mathbf{a}_\phi + 5\mathbf{a}_z$
(e) $5\mathbf{a}_\phi + 3\mathbf{a}_z$

2.7 Given $\mathbf{G} = 20\mathbf{a}_r + 50\mathbf{a}_\theta + 40\mathbf{a}_\phi$, at $(1, \pi/2, \pi/6)$ the component of $\mathbf{G}$ perpendicular to surface $\theta = \pi/2$ is

(a) $20\mathbf{a}_r$
(b) $50\mathbf{a}_\theta$
(c) $40\mathbf{a}_\phi$
(d) $20\mathbf{a}_r + 40\mathbf{a}_\theta$
(e) $-40\mathbf{a}_r + 20\mathbf{a}_\phi$

2.8 Where surfaces $\rho = 2$ and $z = 1$ intersect is

(a) an infinite plane
(b) a semiinfinite plane
(c) a circle
(d) a cylinder
(e) a cone

2.9 Match the items in the left list with those in the right list. Each answer can be used once, more than once, or not at all.

(a) $\theta = \pi/4$	(i) infinite plane
(b) $\phi = 2\pi/3$	(ii) semiinfinite plane
(c) $x = -10$	(iii) circle
(d) $r = 1, \theta = \pi/3, \phi = \pi/2$	(iv) semicircle
(e) $\rho = 5$	(v) straight line
(f) $\rho = 3, \phi = 5\pi/3$	(vi) cone
(g) $\rho = 10, z = 1$	(vii) cylinder
(h) $r = 4, \phi = \pi/6$	(viii) sphere
(i) $r = 5, \theta = \pi/3$	(ix) cube
	(x) point

2.10 A wedge is described by $z = 0, 30° < \phi < 60°$. Which of the following is incorrect:

(a) The wedge lies in the $x - y$ plane.

(b) It is infinitely long

(c) On the wedge, $0 < \rho < \infty$

(d) A unit normal to the wedge is $\pm \mathbf{a}_z$

(e) The wedge includes neither the x-axis nor the y-axis