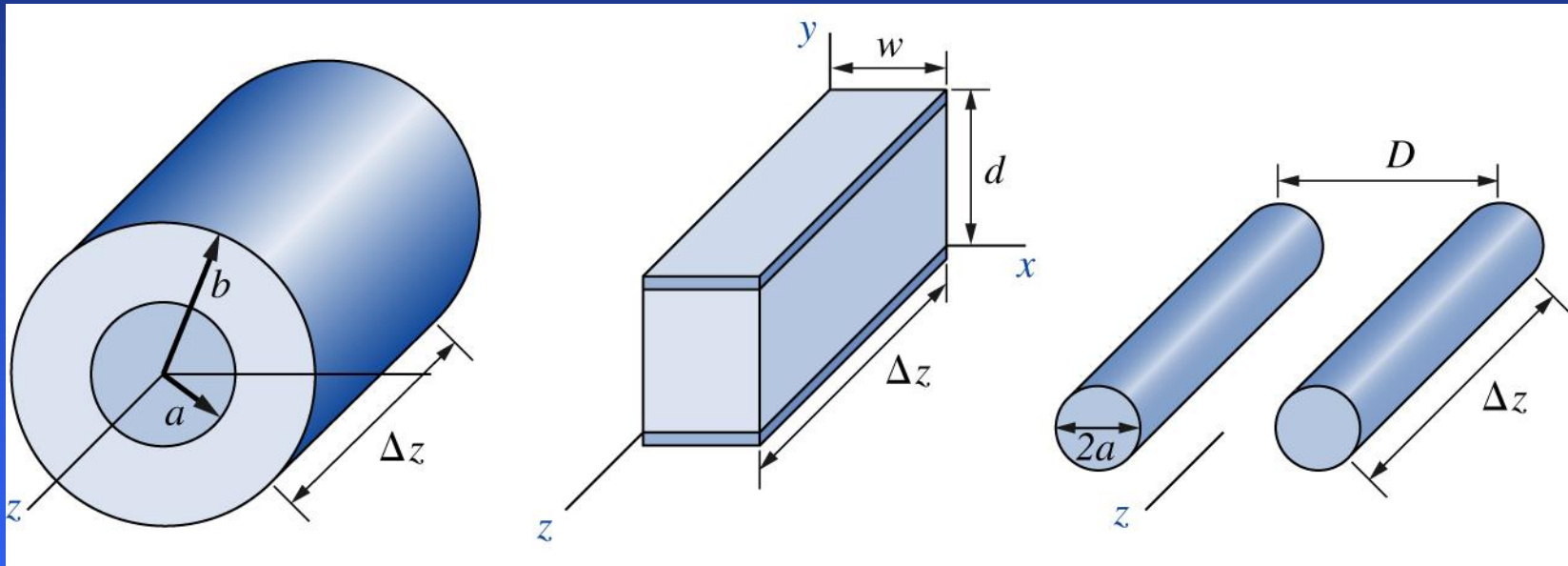


EM Transmission Lines and Smith Chart



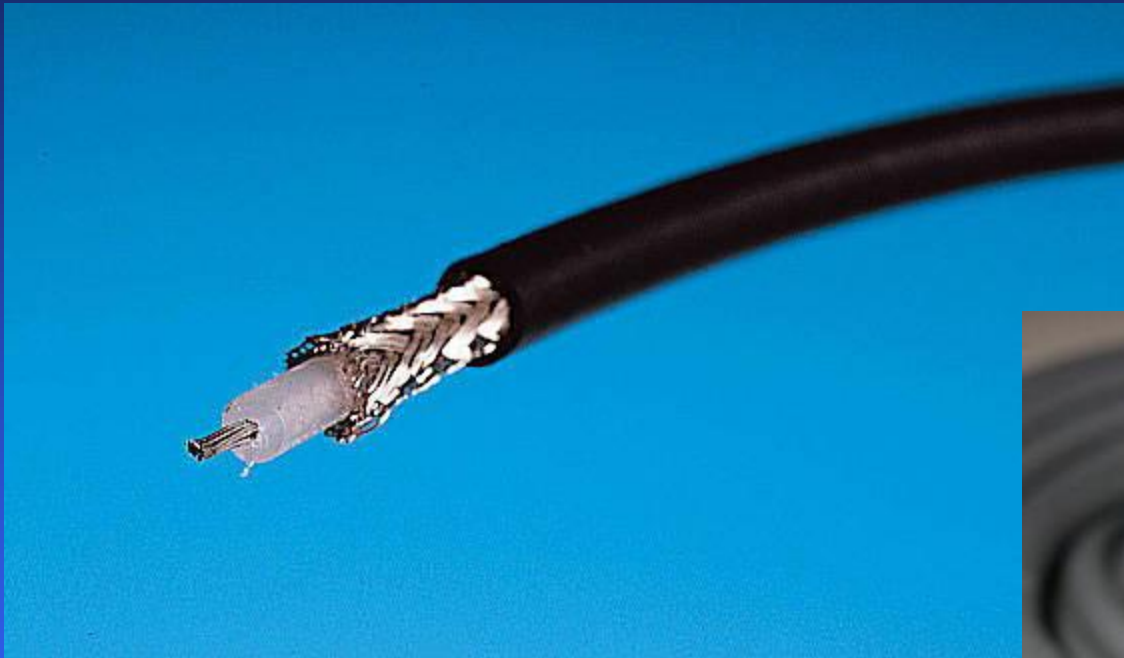
Equivalent electrical circuits

In this topic, we model three electrical transmission systems that can be used to transmit power: a coaxial cable, a strip line, and two parallel wires (twin lead).

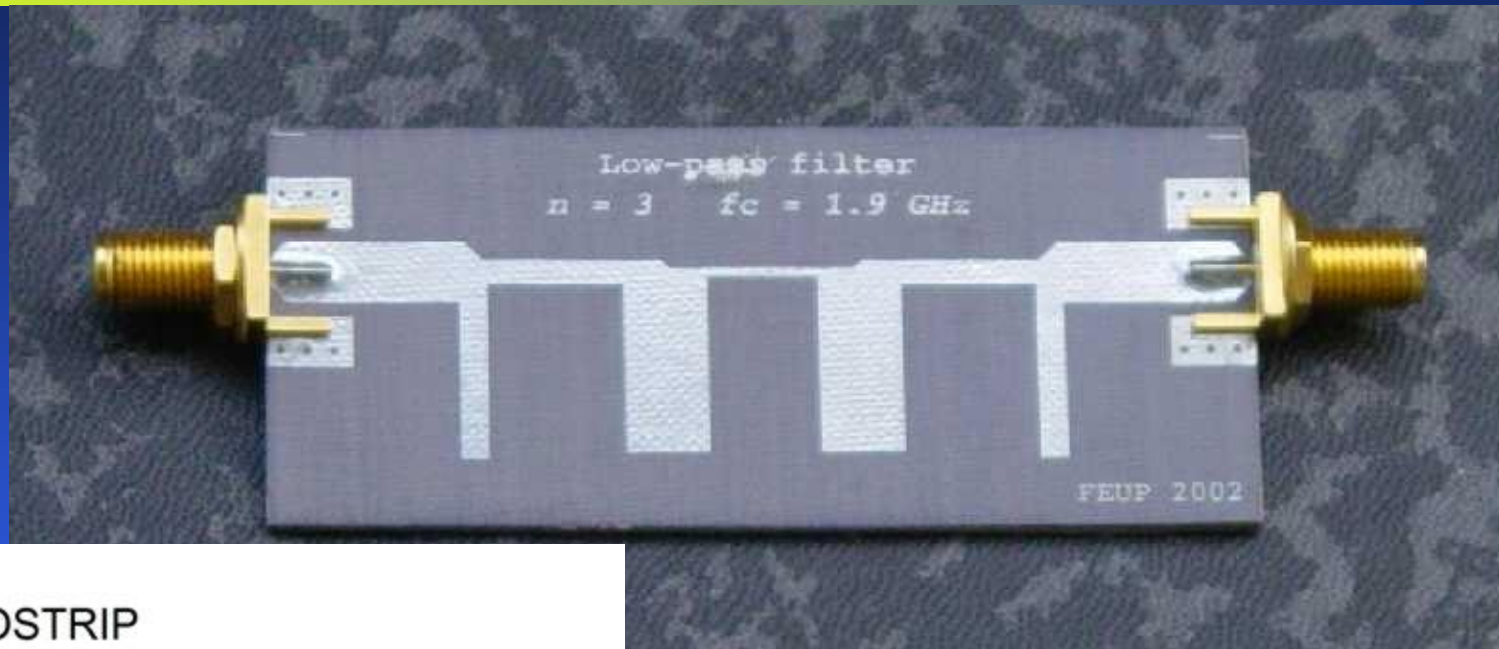


Each structure (including the twin lead) may have a dielectric between two conductors used to keep the separation between the metallic elements constant, so that the electrical properties would be constant.

Coaxial cable

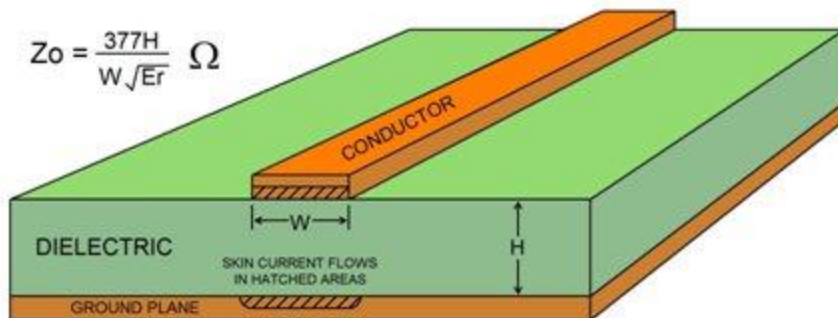


Microstrip line

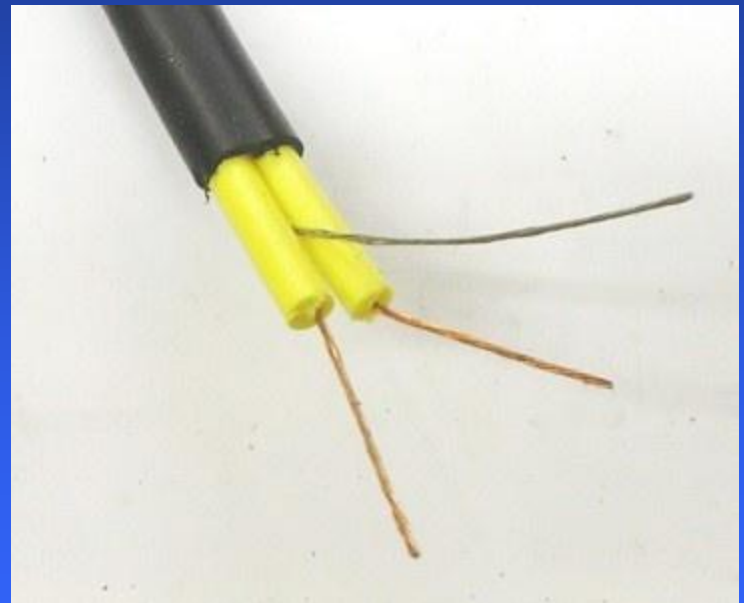
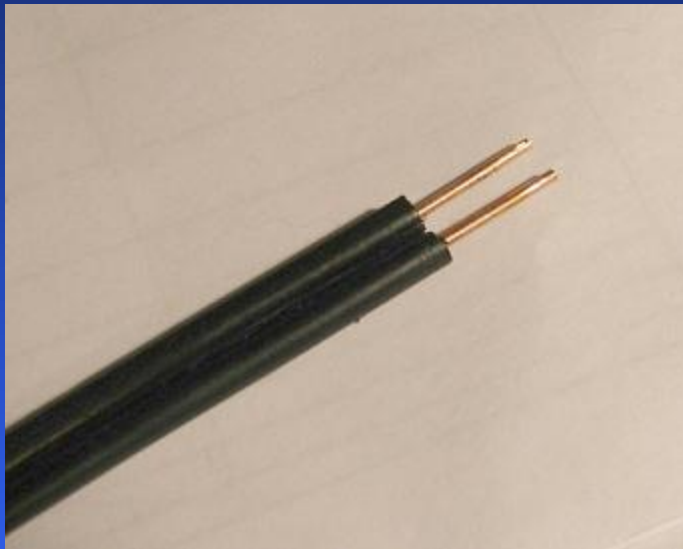


MICROSTRIP TRANSMISSION LINE

$$Z_0 = \frac{377H}{W\sqrt{\epsilon_r}} \Omega$$



Twin lead



Equivalent electrical circuits

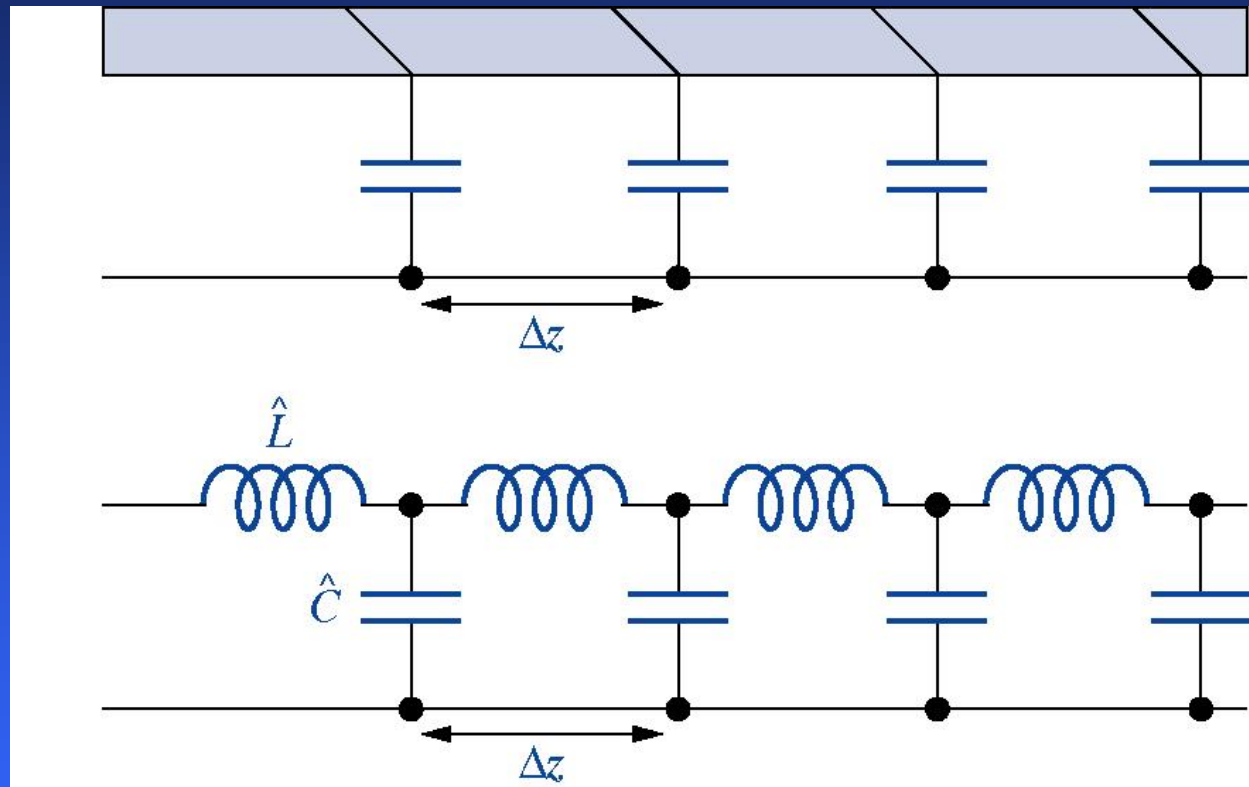
Instead of examining the EM field distribution within these transmission lines, we will simplify our discussion by using a simple model consisting of distributed inductors and capacitors. This model is valid if any dimension of the line transverse to the direction of propagation is much less than the wavelength in a free space.

The transmission lines considered here support the propagation of waves having both electric and magnetic field intensities transverse to the direction of wave propagation. This setup is sometimes called a transverse electromagnetic (TEM) mode of propagation. We assume no loss in the lines.



Equivalent electrical circuits

Distributed transmission line



Its equivalent circuit

Δz is a short distance containing the distributed circuit parameter.

\hat{L} and \hat{C} are distributed inductance and distributed capacitance.

Therefore, each section has inductance $L = \hat{L}\Delta z$ and capacitance $C = \hat{C}\Delta z$ (9.7.1)

Equivalent electrical circuits

Line	Inductance	Capacitance	
Coaxial cable	$L = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) \Delta z$	$C = \frac{2\pi\epsilon}{\ln(b/a)} \Delta z$	(9.8.1)
Microstrip line	$L = \frac{\mu d}{w} \Delta z$	$C = \frac{\epsilon w}{d} \Delta z$	(9.8.2)
Twin lead	$L = \frac{\mu}{\pi} \cosh^{-1}\left(\frac{D}{2a}\right) \Delta z$	$C = \frac{\pi\epsilon}{\cosh^{-1}(D/2a)} \Delta z$	(9.8.3)

Note: the equations for a microstrip line are simplified and do not include effects of fringing.

We can model the transmission line with an equivalent circuit consisting of an infinite number of distributed inductors and capacitors.

Equivalent electrical circuits

The following simplifications were used:

- 1) No energy loss (resistance) was incorporated;
- 2) We neglected parasitic capacitances between the wires that constitute the distributed inductances. We will see later that these parasitic capacitances will lead to changes in phase velocity of the wave (dispersion);
- 3) Parameters of the line are constant.

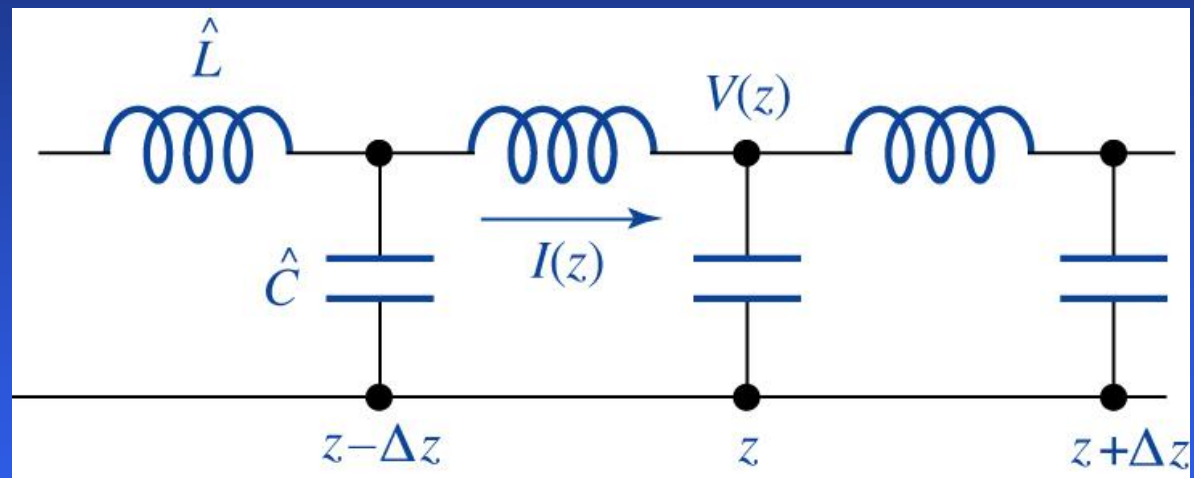
We can analyze EM transmission lines either as a large number of distributed two-port networks or as a coupled set of first-order PDEs that are called the telegraphers' equations.



Transmission line equations

While analyzing the equivalent circuit of the lossless transmission line, it is simpler to use Kirchhoff's laws rather than Maxwell's equations.

Therefore, we will consider the equivalent circuit of this form:



For simplicity, we define the inductance and capacitance per unit length:

$$\hat{L} = \frac{L}{\Delta z}; \quad \hat{C} = \frac{C}{\Delta z} \quad (9.10.1)$$

which have units of Henries per unit length and Farads per unit length, respectively.

Transmission line equations

The current entering the node at the location z is $I(z)$. The part of this current will flow through the capacitor, and the rest flows into the section. Therefore:

$$I(z, t) = \hat{C} \Delta z \frac{\partial V(z, t)}{\partial t} + I(z + \Delta z, t) \quad (9.11.1)$$

$$\frac{I(z + \Delta z, t) - I(z, t)}{\Delta z} = -\hat{C} \frac{\partial V(z, t)}{\partial t} \quad (9.11.2)$$

If $\Delta z \rightarrow 0$, the LHS of (9.11.2) is a spatial derivative. Therefore:

$$\frac{\partial I(z, t)}{\partial z} = -\hat{C} \frac{\partial V(z, t)}{\partial t} \quad (9.11.3)$$

Transmission line equations

Similarly, the sum of the voltage drops in this section can be calculated via the Kirchhoff's law also:

$$V(z - \Delta z, t) = \hat{L} \Delta z \frac{\partial I(z, t)}{\partial t} + V(z, t) \quad (9.12.1)$$

$$\frac{V(z, t) - V(z - \Delta z, t)}{\Delta z} = -\hat{L} \frac{\partial I(z, t)}{\partial t} \quad (9.12.2)$$

If $\Delta z \rightarrow 0$, the LHS of (9.12.2) is a spatial derivative. Therefore:

$$\frac{\partial V(z, t)}{\partial z} = -\hat{L} \frac{\partial I(z, t)}{\partial t} \quad (9.12.3)$$

Transmission line equations

The equations (9.11.3) and (9.12.3) are two linear coupled first-order PDEs called the telegrapher's (Heaviside) equations. They can be composed in a second-order PDE:

$$\frac{\partial^2 I(z,t)}{\partial z^2} - \hat{L}\hat{C} \frac{\partial^2 I(z,t)}{\partial t^2} = 0 \quad (9.13.1)$$

$$\frac{\partial^2 V(z,t)}{\partial z^2} - \hat{L}\hat{C} \frac{\partial^2 V(z,t)}{\partial t^2} = 0 \quad (9.13.2)$$

We may recognize that both (9.13.1) and (9.13.2) are wave equations with the velocity of propagation:

$$v = \frac{1}{\sqrt{\hat{L}\hat{C}}} \quad (9.13.3)$$

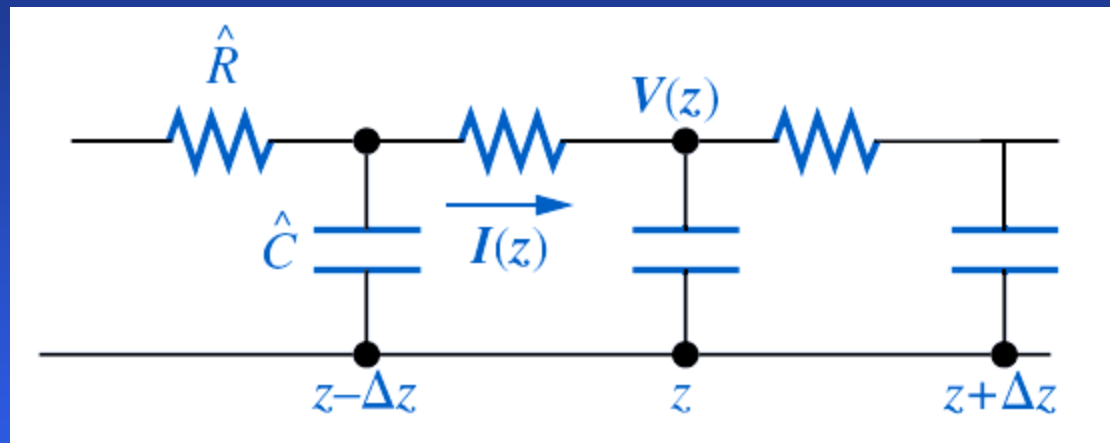


Transmission line equations

Example 9.1: Show that a transmission line consisting of distributed linear resistors and capacitors in the given configuration can be used to model diffusion.

We assume that the resistance and the capacitance per unit length are defined as

$$\hat{R} = \frac{R}{\Delta z}; \quad \hat{C} = \frac{C}{\Delta z} \quad (9.14.1)$$



Potential drop over the resistor R and the current through the capacitor C are:

$$\Delta V(z, t) = I(z, t) \hat{R} \Delta z \quad (9.14.2)$$

$$\Delta I(z, t) = \hat{C} \Delta z \frac{\partial V(z, t)}{\partial t} \quad (9.14.3)$$

Transmission line equations

$$\text{If } \Delta z \rightarrow 0 \Rightarrow \frac{\partial V(z,t)}{\partial z} = I(z,t) \hat{R} \quad (9.15.1)$$

$$\frac{\partial I(z,t)}{\partial z} = \hat{C} \frac{\partial V(z,t)}{\partial t} \quad (9.15.2)$$

The corresponding second-order PDE for the potential is:

$$\frac{\partial^2 V(z,t)}{\partial z^2} = \hat{R} \frac{\partial I(z,t)}{\partial z} = \hat{R} \left(\hat{C} \frac{\partial V(z,t)}{\partial t} \right) = \hat{R} \hat{C} \frac{\partial V(z,t)}{\partial t} \quad (9.15.3)$$

Which is a form of a diffusion equation with a diffusion coefficient:

$$D = \frac{1}{\hat{R} \hat{C}} \quad (9.15.4)$$



Transmission line equations

Example 9.2: Show that a particular solution for the diffusion equation is given by

$$V(z, t) = \frac{1}{2\sqrt{D\pi t}} e^{-\frac{z^2}{4Dt}} \quad (9.16.1)$$

Differentiating the solution with respect to z :

$$\frac{\partial V(z, t)}{\partial z} = \frac{1}{2\sqrt{D\pi}} \left(-\frac{z}{2Dt^{3/2}} \right) e^{-\frac{z^2}{4Dt}} \quad (9.16.2)$$

$$\frac{\partial^2 V(z, t)}{\partial z^2} = \frac{1}{2\sqrt{D\pi}} \left(-\frac{z}{2Dt^{3/2}} + \frac{z^2}{4D^2 t^{5/2}} \right) e^{-\frac{z^2}{4Dt}} \quad (9.16.3)$$

Differentiating the solution with respect to t :

$$\frac{1}{D} \frac{\partial V(z, t)}{\partial t} = \frac{1}{D} \frac{1}{2\sqrt{D\pi}} \left(-\frac{t}{2t^{3/2}} + \frac{z^2}{4Dt^{5/2}} \right) e^{-\frac{z^2}{4Dt}} \quad (9.16.4)$$



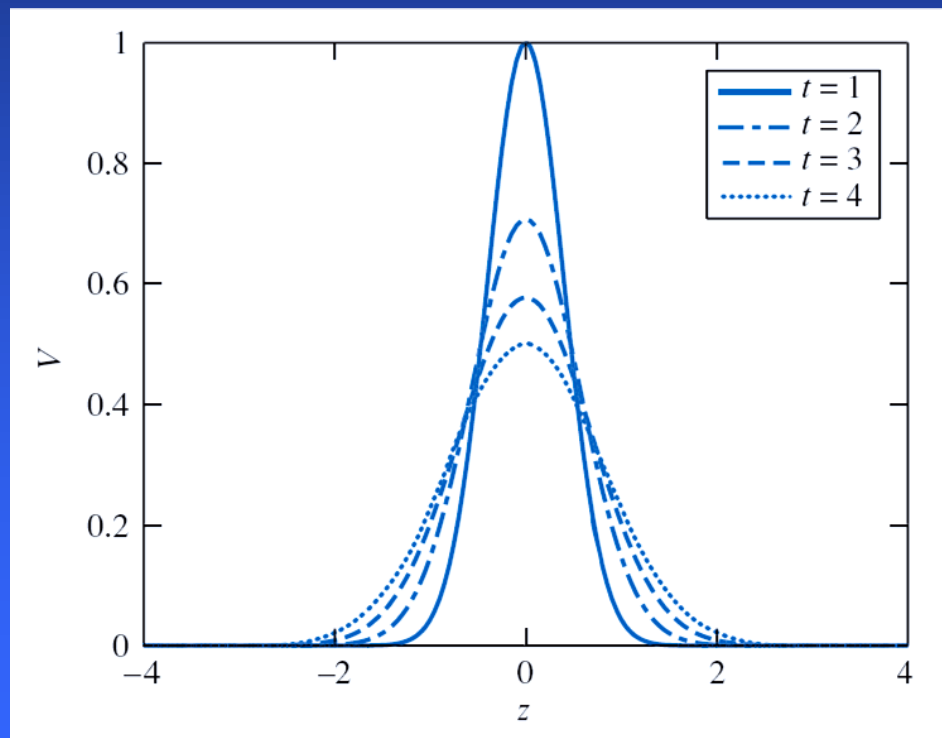
Transmission line equations

Since the RHSs of (9.16.3) and (9.16.4) are equal, the diffusion equation is satisfied.

The voltages at different times are shown. The total area under each curve equals 1.

This solution would be valid if a certain amount of charge is placed at $z = 0$ at some moment in the past.

Note: the diffusion is significantly different from the wave propagation.



Sinusoidal waves

We are looking for the solutions of wave equations (9.13.1) and (9.13.2) for the time-harmonic (AC) case. We must emphasize that – unlike the solution for a static DC case or quasi-static low-frequency case (ones considered in the circuit theory) – these solutions will be in form of traveling waves of voltage and current, propagating in either direction on the transmission line with the velocity specified by (9.13.3).

We assume here that the transmission line is connected to a distant generator that produces a sinusoidal signal at fixed frequency $\omega = 2\pi f$. Moreover, the generator has been turned on some time ago to ensure that transient response decayed to zero; therefore, the line is in a steady-state mode.

The most important (and traditional) simplification for the time-harmonic case is the use of phasors. We emphasize that while in the AC circuits analysis phasors are just complex numbers, for the transmission lines, phasors are complex functions of the position z on the line.



Sinusoidal waves

$$V(z, t) = \text{Re} \{ V(z) e^{j\omega t} \}; \quad I(z, t) = \text{Re} \{ I(z) e^{j\omega t} \} \quad (9.19.1)$$

Therefore, the wave equations will become:

$$\frac{d^2 V(z)}{dz^2} + k^2 V(z) = 0 \quad (9.19.2)$$

$$\frac{d^2 I(z)}{dz^2} + k^2 I(z) = 0 \quad (9.19.3)$$

Here, as previously, k is the wave number:

$$k = \frac{\omega}{v} = \frac{2\pi}{\lambda} \quad (9.19.4)$$

Velocity of propagation \nearrow
 Wavelength of the voltage or current wave



Sinusoidal waves

A solution for the wave equation (9.13.3) can be found, for instance, in one of these forms:

$$V(z) = A_1 \cos kz + B_1 \sin kz \quad (9.20.1)$$

$$V(z) = A_2 e^{-jkz} + B_2 e^{+jkz} \quad (9.20.2)$$

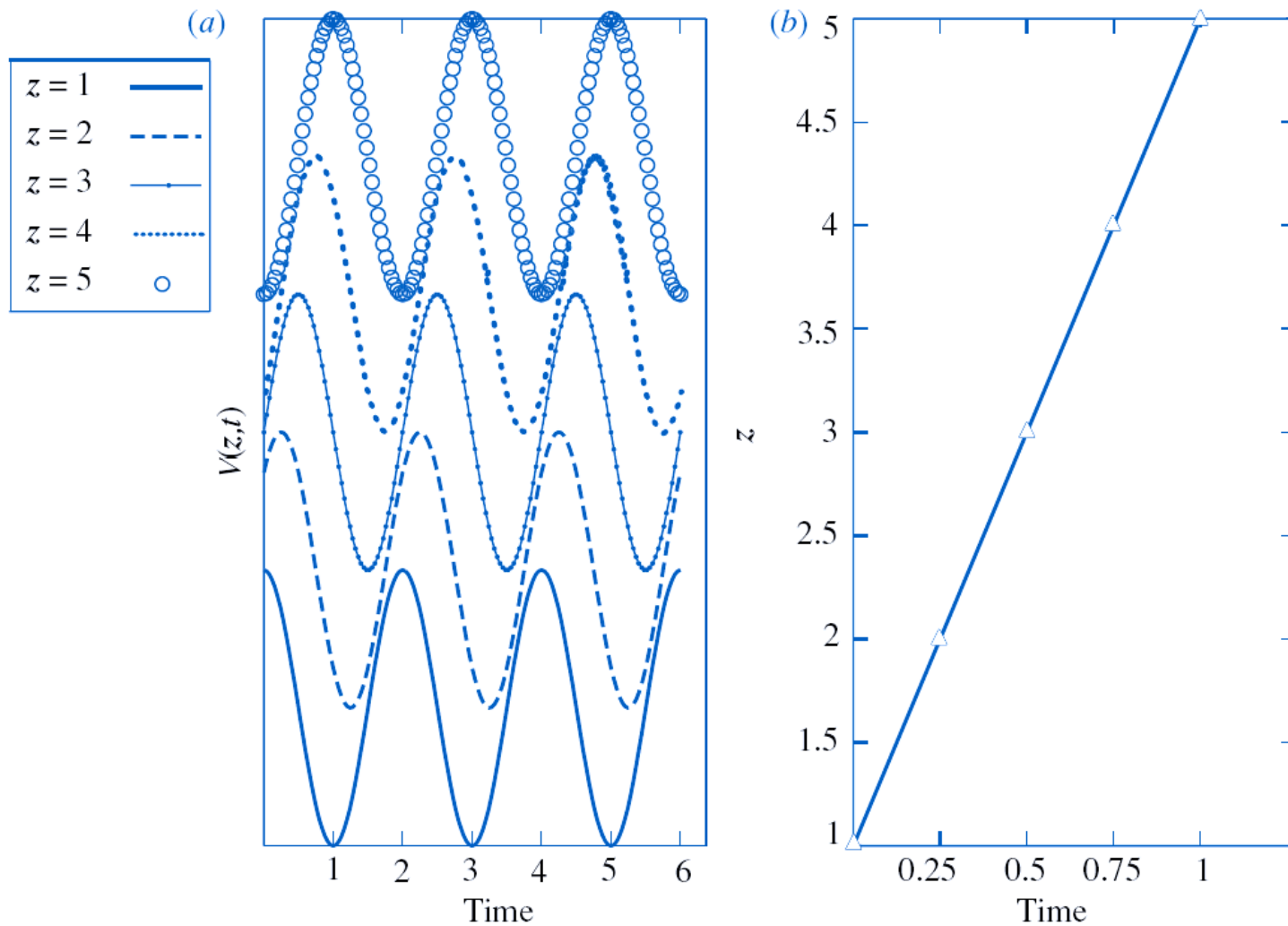
We select the exponential form (9.20.2) since it is easier to interpret in terms of propagating waves of voltage on the transmission line.

Example 9.3: The voltage of a wave propagating through a transmission line was continuously measured by a set of detectors placed at different locations along the transmission line. The measured values are plotted. Write an expression for the wave for the given data.



Sinusoidal waves

The data



Slope of the trajectory...

Sinusoidal waves

We assume that the peak-to-peak amplitude of the wave is $2V_0$. We also conclude that the wave propagates in the $+z$ direction.

The period of the wave is 2s, therefore, the frequency of oscillations is $\frac{1}{2}$ Hz.

$$\omega = 2\pi f = \pi \quad (9.22.1)$$

The velocity of propagation can be found from the slope as:

$$v = \frac{5-1}{1-0} = 4 \text{ m/s} \quad (9.22.2)$$

The wave number is:

$$k = \frac{\omega}{v} = \frac{2\pi \cdot 0.5}{4} = \frac{\pi}{4} \text{ m}^{-1} \quad (9.22.3)$$

The wavelength is:

$$\lambda = \frac{2\pi}{k} = \frac{4 \cdot 2\pi}{\pi} = 8 \text{ m} \quad (9.22.4)$$

← Not in vacuum!

Therefore, the wave is:

$$V(z, t) = V_0 e^{j\left(\pi t - \frac{\pi z}{4}\right)} \quad (9.22.5)$$

Sinusoidal waves

Assuming next that the source is located far from the observation point (say, at $z = -\infty$) and that the transmission line is infinitely long, there would be only a forward traveling wave of voltage on the transmission line. In this case, the voltage on the transmission line is:

$$V(z) = V_0 e^{-jkz} \quad (9.23.1)$$

The phasor form of (9.12.3) in this case is

$$\frac{dV(z)}{dz} = -jkV(z) = -j\omega\hat{L}I(z) \quad (9.23.2)$$

Which may be rearranged as:

$$I(z) = \frac{k}{\omega\hat{L}}V(z) = \frac{k}{\omega\hat{L}}V_0 e^{-jkz} \quad (9.23.3)$$



Sinusoidal waves

The ratio of the voltage to the current is a very important transmission line parameter called the characteristic impedance:

$$Z_c = \frac{V(z)}{I(z)} = \frac{\omega \hat{L}}{k} \quad (9.24.1)$$

Since $k = \frac{\omega}{v}$ and $v = \frac{1}{\sqrt{\hat{L}\hat{C}}}$ (9.24.2)

Then: $Z_c = \sqrt{\frac{\hat{L}}{\hat{C}}} \text{ } [\Omega] \quad (9.24.3)$

We emphasize that (9.24.3) is valid for the case when only one wave (traveling either forwards or backwards) exists. In a general case, more complicated expression must be used. If the transmission line was lossy, the characteristic impedance would be complex.



Sinusoidal waves

If we know the characteristics of the transmission line and the forward voltage wave, we may find the forward current wave by dividing voltage by the characteristic impedance.

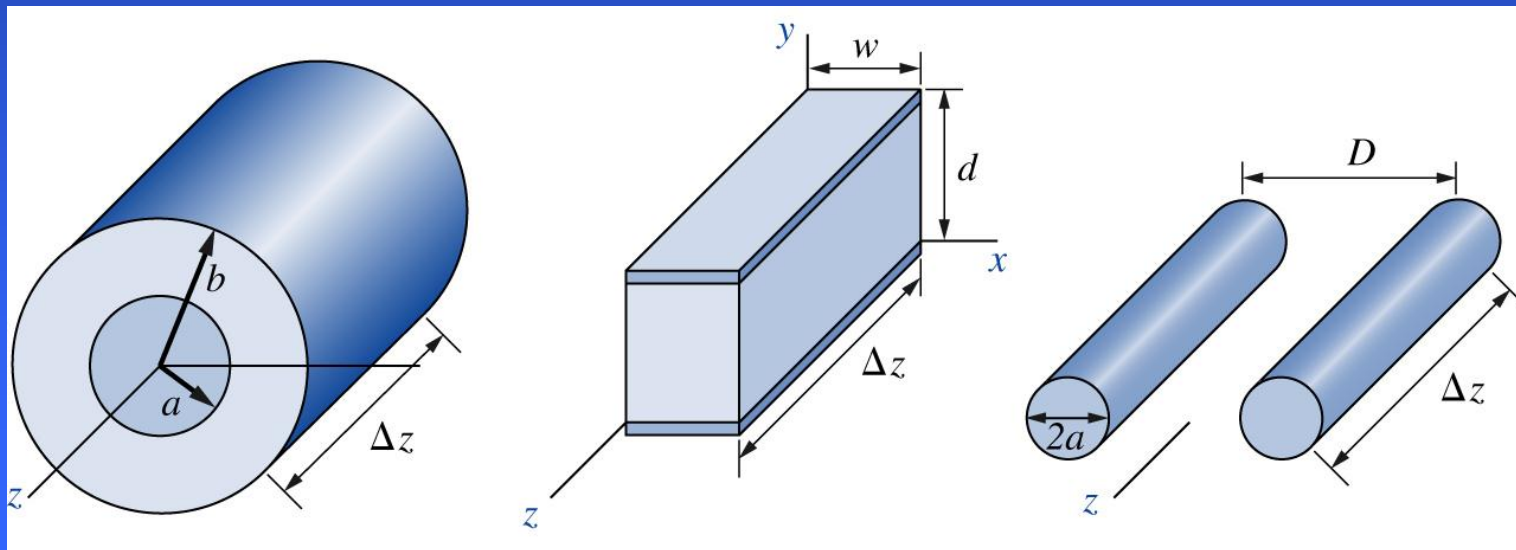
Another important parameter of a transmission line is its length L , which is often normalized by the wavelength of the propagation wave.

Assuming that the dielectric between conductors has ϵ and μ

Line	Velocity of propagation	Characteristic impedance
Coaxial cable	$v = \frac{1}{\sqrt{\hat{L}\hat{C}}} = \frac{1}{\sqrt{\mu\epsilon}}$	$Z_c = \sqrt{\frac{\hat{L}}{\hat{C}}} = \sqrt{\frac{\mu}{\epsilon}} \left(\frac{\ln(b/a)}{2\pi} \right)$
Microstrip line	$v = \frac{1}{\sqrt{\hat{L}\hat{C}}} = \frac{1}{\sqrt{\mu\epsilon}}$	$Z_c = \sqrt{\frac{\hat{L}}{\hat{C}}} = \sqrt{\frac{\mu}{\epsilon}} \left(\frac{d}{w} \right)$
Twin lead	$v = \frac{1}{\sqrt{\hat{L}\hat{C}}} = \frac{1}{\sqrt{\mu\epsilon}}$	$Z_c = \sqrt{\frac{\hat{L}}{\hat{C}}} = \sqrt{\frac{\mu}{\epsilon}} \left(\frac{\cosh^{-1}(D/2a)}{\pi} \right)$

Sinusoidal waves

The velocity of propagation does not depend on the dimensions of the transmission line and is only a function of the parameters of the material that separates two conductors. However, the characteristic impedance DOES depend upon the geometry and physical dimensions of the transmission line.



Sinusoidal waves

Example 9.4: Evaluate the velocity of propagation and the characteristic impedance of an air-filled coaxial cable with radii of the conductors of 3 mm and 6 mm.

The inductance and capacitance per unit length are:

$$\hat{L} = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) = \frac{4\pi \cdot 10^{-7}}{2\pi} \ln\left(\frac{6}{3}\right) = 0.14 \text{ } \mu\text{H}/\text{m} \quad (9.27.1)$$

$$\hat{C} = \frac{2\pi\epsilon_0}{\ln(b/a)} = \frac{2\pi \cdot 8.854 \cdot 10^{-12}}{\ln(6/3)} = 80 \text{ } \text{pF}/\text{m} \quad (9.27.2)$$

The velocity of propagation is:

$$v = \frac{1}{\sqrt{\hat{L}\hat{C}}} = \frac{1}{\sqrt{0.14 \cdot 10^{-6} \cdot 80 \cdot 10^{-12}}} = 3 \cdot 10^8 \text{ } \text{m}/\text{s} \quad (9.27.3)$$

The characteristic impedance of the cable is:

$$Z_c = \sqrt{\hat{L}/\hat{C}} = \sqrt{0.14 \cdot 10^{-6} / 80 \cdot 10^{-12}} \approx 42 \text{ } \Omega \quad (9.27.4)$$

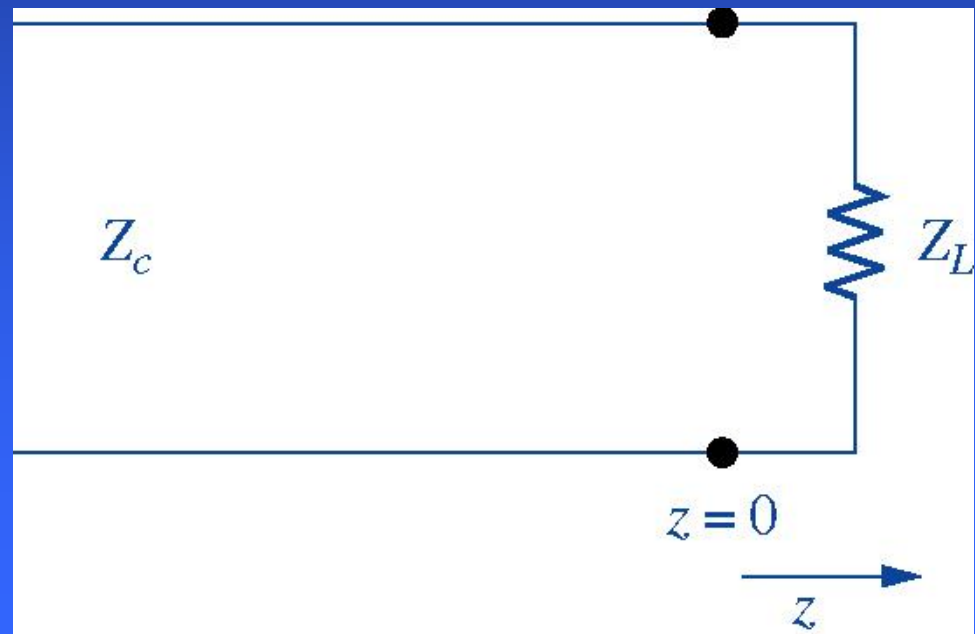
Both the v and the Z_c may be decreased by insertion of a dielectric between leads.



Terminators

So far, we assumed that the transmission line was infinite. In the reality, however, transmission lines have both the beginning and the end.

The line has a real characteristic impedance Z_c . We assume that the source of the wave is at $z = -\infty$ and the termination (the end of the line) is at $z = 0$. The termination may be either an impedance or another transmission line with different parameters. We also assume no transients.



Terminators

The phasor voltage at any point on the line is:

$$V(z) = A_2 e^{-jkz} + B_2 e^{+jkz} \quad (9.29.1)$$

The phasor current is:

$$I(z) = \frac{A_2 e^{-jkz} - B_2 e^{+jkz}}{Z_c} \quad (9.29.2)$$

At the load location ($z = 0$), the ratio of voltage to current must be equal Z_L :

$$Z_L = \frac{V(z=0)}{I(z=0)} = Z_c \frac{A_2 + B_2}{A_2 - B_2} \quad (9.29.3)$$

Note: the ratio B_2 to A_2 represents the magnitude of the wave incident on the load Z_L .



Terminators

We introduce the reflection coefficient for the transmission line with a load as:

$$\Gamma \equiv \frac{B_2}{A_2} = \frac{Z_L - Z_c}{Z_L + Z_c} \quad (9.30.1)$$

Often, the normalized impedance is used:

$$z_L = \frac{Z_L}{Z_c} \quad (9.30.2)$$

The reflection coefficient then becomes:

$$\Gamma \equiv \frac{B_2}{A_2} = \frac{z_L - 1}{z_L + 1} \quad (9.30.3)$$



Terminators

Therefore, the phasor representations for the voltage and the current are:

$$V(z) = V_0 \left[e^{-jkz} + \Gamma e^{+jkz} \right] \quad (9.31.1)$$

$$I(z) = \frac{V_0}{Z_c} \left[e^{-jkz} - \Gamma e^{+jkz} \right] \quad (9.31.2)$$

The total impedance is:

$$Z(z) = \frac{V(z)}{I(z)} \quad (9.31.3)$$

generally a complicated function of the position and NOT equal to Z_c . However, a special case of matched load exists when:

$$Z_L = Z_c \quad (9.31.4)$$

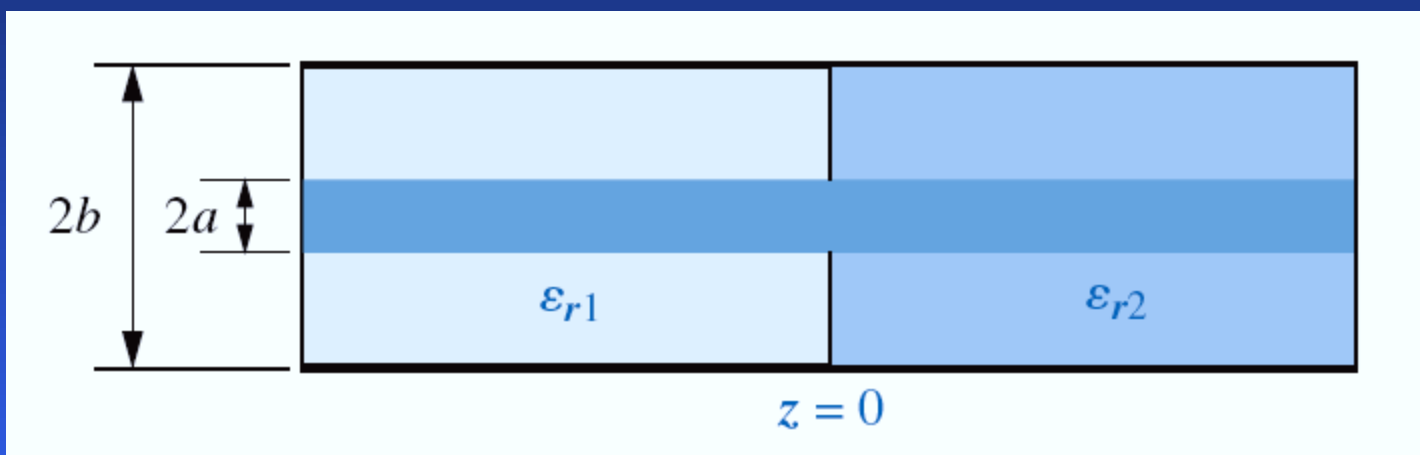
In this situation:

$$Z(z) = Z_c \quad (9.31.5)$$



Terminators

Example 9.5: Evaluate the reflection coefficient for a wave that is incident from $z = -\infty$ in an infinitely long coaxial cable that has $\epsilon_r = 2$ for $z < 0$ and $\epsilon_r = 3$ for $z > 0$.



The characteristic impedance is:

$$Z_c = \sqrt{\frac{\hat{L}}{\hat{C}}} = \sqrt{\frac{\mu}{\epsilon} \frac{\ln(b/a)}{2\pi}}$$

The load impedance of a line is the characteristic impedance of the line for $z > 0$.

Terminators

The reflection coefficient can be expressed as:

$$\Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{\sqrt{\frac{\mu_0}{\epsilon_{r2}\epsilon_0}} \left(\frac{\ln[b/a]}{2\pi} \right) - \sqrt{\frac{\mu_0}{\epsilon_{r1}\epsilon_0}} \left(\frac{\ln[b/a]}{2\pi} \right)}{\sqrt{\frac{\mu_0}{\epsilon_{r2}\epsilon_0}} \left(\frac{\ln[b/a]}{2\pi} \right) + \sqrt{\frac{\mu_0}{\epsilon_{r1}\epsilon_0}} \left(\frac{\ln[b/a]}{2\pi} \right)}$$

$$= \frac{\sqrt{\frac{1}{\epsilon_{r2}}} - \sqrt{\frac{1}{\epsilon_{r1}}}}{\sqrt{\frac{1}{\epsilon_{r2}}} + \sqrt{\frac{1}{\epsilon_{r1}}}} = \frac{\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{2}}} \approx -0.1$$



Terminators

The reflection coefficient Γ is completely determined by the value of the impedance of the load and the characteristic impedance of the transmission line. The reflection coefficient for a lossless transmission line can have any complex value with magnitude less or equal to one.

If the load is a short circuit ($Z_L = 0$), the reflection coefficient $\Gamma = -1$. The voltage at the load is a sum of voltages of the incident and the reflected components and must be equal to zero since the voltage across the short circuit is zero.

If the load is an open circuit ($Z_L = \infty$), the reflection coefficient $\Gamma = +1$. The voltage at the load can be arbitrary but the total current must be zero.

If the load impedance is equal to the characteristic impedance ($Z_L = Z_c$), the reflection coefficient $\Gamma = 0$ – line is **matched**. In this case, all energy of generator will be absorbed by the load.



Terminators

For the shorten transmission line:

$$\Gamma = -1 \quad (9.35.1)$$

$$V(z,t) = \text{Re} \left\{ V_0 \left[e^{-jkz} - e^{+jkz} \right] e^{j\omega t} \right\}$$

$$= 2V_0 \sin kz \cdot \cos \left(\omega t - \frac{\pi}{2} \right) \quad (9.35.2)$$

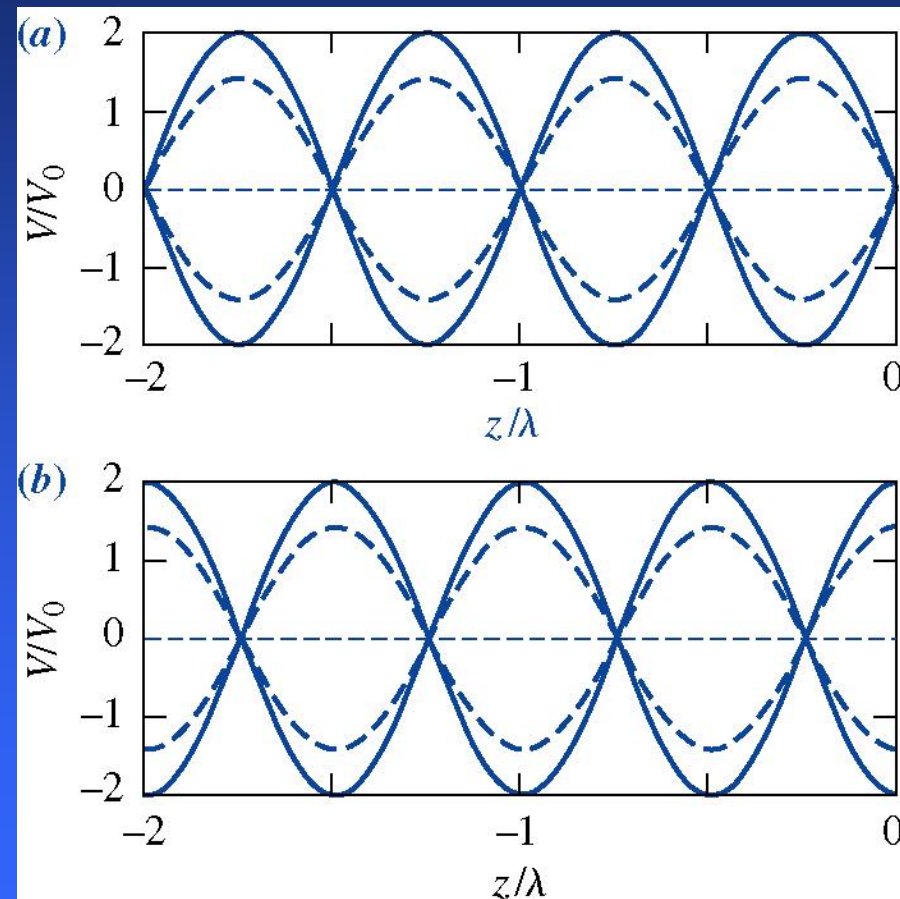
For the open transmission line:

$$\Gamma = +1 \quad (9.35.3)$$

$$V(z,t) = \text{Re} \left\{ V_0 \left[e^{-jkz} + e^{+jkz} \right] e^{j\omega t} \right\}$$

$$= 2V_0 \cos kz \cdot \cos(\omega t) \quad (9.35.4)$$

In both cases, a standing wave is created.
The signal does not appear to propagate.

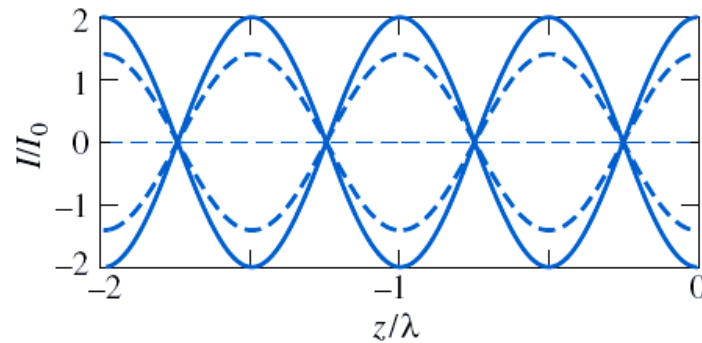
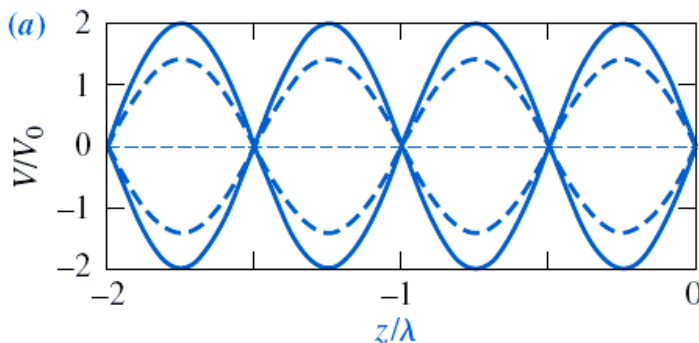


Terminators

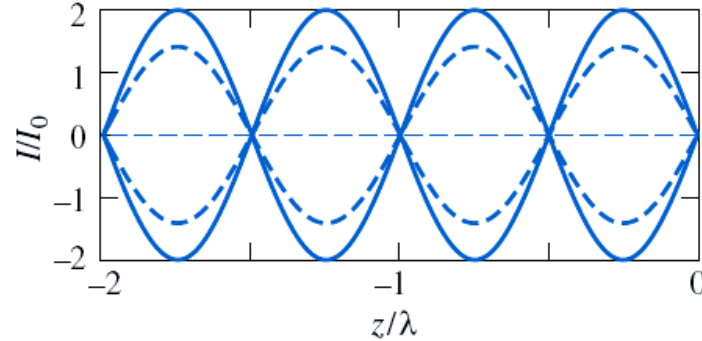
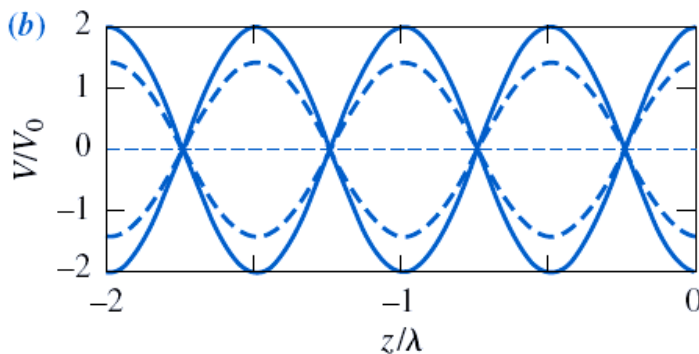
Since the current can be found as

$$I_0 = V_0/Z_c$$

$$Z_L = 0$$



$$Z_L = \infty$$



Note that the current wave differs from the voltage wave by 90°

Terminators

Another important quantity is the ratio of the maximum voltage to the minimum voltage called the voltage standing wave ratio:

$$VSWR = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \quad (9.37.1)$$

Which leads to

$$|\Gamma| = \frac{VSWR - 1}{VSWR + 1} \quad (9.37.2)$$

VSWR, the reflection coefficient, the load impedance, and the characteristic impedance are related.

Even when the amplitude of the incident wave V_0 does not exceed the maximally allowed value for the transmission line, reflection may lead to the voltage $V_0(1+|\Gamma|)$ exceeding the maximally allowed.

Therefore, the load and the line must be matched.

Terminators

Example 9.6: Evaluate the *VSWR* for the coaxial cable described in the Example 9.5. The reflection coefficient was evaluated as $\Gamma = -0.1$.

$$VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + |-0.1|}{1 - |-0.1|} = 1.2$$

Note: if two cables were matched, the *VSWR* would be 1.



Terminators

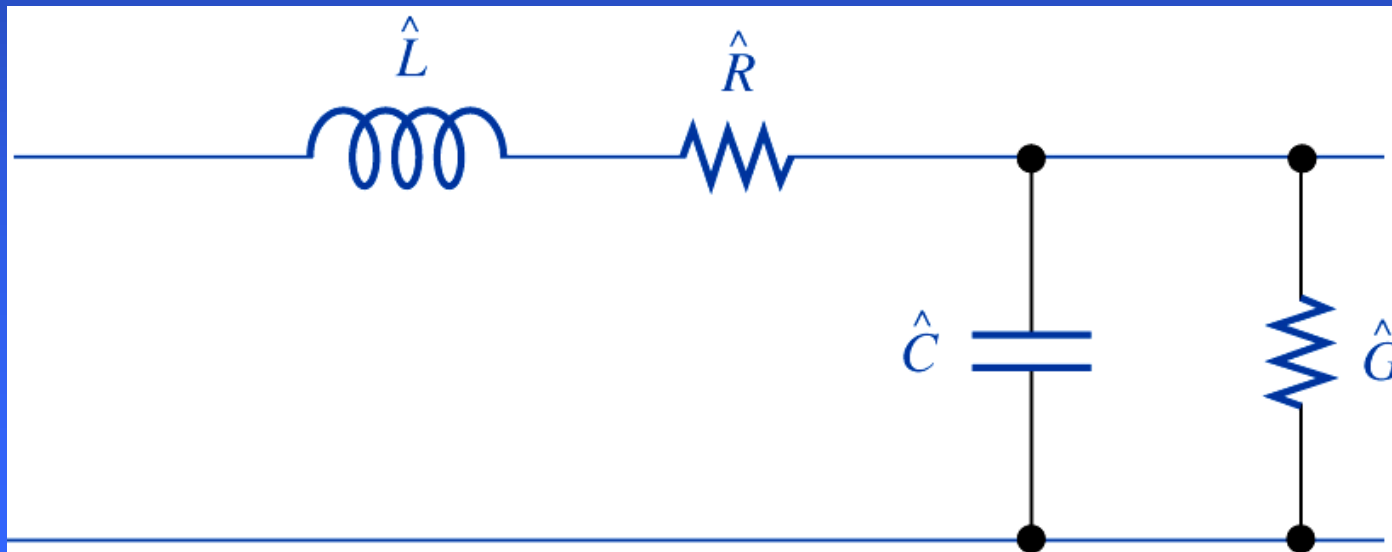


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Lossy transmission lines

Up to this point, our discussion was limited to lossless transmission lines consisting of equivalent inductors and capacitors only. As a result, the characteristic impedance for such lines is real. Let us incorporate ohmic losses within the conductors and leakage currents between conductors. In this case, the characteristic impedance becomes complex and the new model will be:



Lossy transmission lines

The new first-order PDEs (telegrapher's equations) are

$$\frac{\partial I(z,t)}{\partial z} = -\hat{C} \frac{\partial V(z,t)}{\partial t} - \hat{G}V(z,t) \quad (9.83.1)$$

$$\frac{\partial V(z,t)}{\partial z} = -\hat{L} \frac{\partial I(z,t)}{\partial t} - \hat{R}I(z,t) \quad (9.83.2)$$

Where the circuit elements are defines as

$$\hat{L} = \frac{L}{\Delta z}; \quad \hat{C} = \frac{C}{\Delta z}; \quad \hat{R} = \frac{R}{\Delta z}; \quad \hat{G} = \frac{G}{\Delta z} \quad (9.83.3)$$

A time-harmonic excitation of the transmission line leads to a phasor notation:

$$\frac{\partial I(z)}{\partial z} = -\left[\hat{G} + j\omega\hat{C}\right]V(z) \quad (9.83.4)$$

$$\frac{\partial V(z)}{\partial z} = -\left[\hat{R} + j\omega\hat{L}\right]I(z) \quad (9.83.5)$$

Lossy transmission lines

The quantities in square brackets are denoted by distributed admittance \hat{Y} and distributed impedance \hat{Z} leading to

$$\frac{dI(z)}{dz} = -\hat{Y}V(z) \quad (9.84.1)$$

$$\frac{dV(z)}{dz} = -\hat{Z}I(z) \quad (9.84.2)$$

Second-order ODEs can be derived for current and voltage:

$$\frac{d^2I(z)}{dz^2} = \hat{Z}\hat{Y}I(z) \quad (9.84.3)$$

$$\frac{d^2V(z)}{dz^2} = \hat{Z}\hat{Y}V(z) \quad (9.84.4)$$

The phasor form solutions:

$$V(z) = V_1e^{-\gamma z} + V_2e^{+\gamma z} \quad (9.84.5)$$

$$I(z) = I_1e^{-\gamma z} + I_2e^{+\gamma z} \quad (9.84.6)$$



Lossy transmission lines

Here, γ is the complex propagation constant:

$$\gamma = \alpha + j\beta = \sqrt{\hat{Z}\hat{Y}} = \sqrt{(\hat{R} + j\omega\hat{L})(\hat{G} + j\omega\hat{C})} \quad (9.85.1)$$

As previously, we denote by V_1 and I_1 the amplitudes of the forward (in the +z direction) propagating voltage and current waves; and V_2 and I_2 are the amplitudes of the backward (in the -z direction) propagating voltage and current waves. Therefore, the time varying waves will be in the form:

$$V(z, t) = V_1 e^{-\alpha z} \cos(\omega t - \beta z) + V_2 e^{+\alpha z} \cos(\omega t + \beta z) \quad (9.85.2)$$

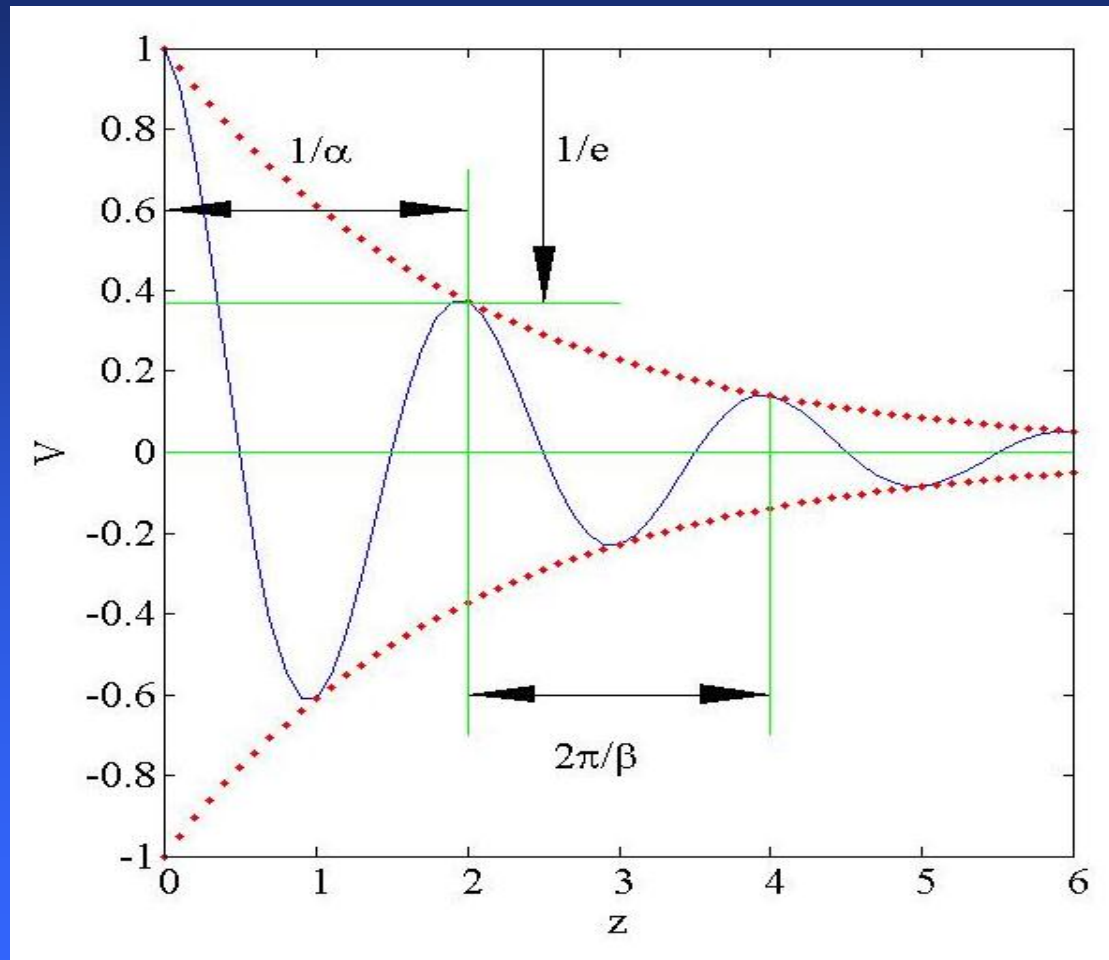
$$I(z, t) = I_1 e^{-\alpha z} \cos(\omega t - \beta z) + I_2 e^{+\alpha z} \cos(\omega t + \beta z) \quad (9.85.3)$$

We recognize that (9.85.2) and (9.85.3) are exponentially decaying propagating waves: the forward wave (V_1, I_1) propagates and decays in the +z direction, while the backward wave (V_2, I_2) propagates and decays in the -z direction.



Lossy transmission lines

An example of a forward propagating wave: it is possible to determine the values of α and β from the graph as shown.



Lossy transmission lines

For small losses, employing a binomial approximation $(1 - x)^n \cong 1 - nx$ for $x \ll 1$, the complex propagating constant is approximately

$$\gamma = \alpha + j\beta = \sqrt{j\omega\hat{L}}\sqrt{j\omega\hat{C}}\sqrt{\left(1 + \frac{\hat{R}}{j\omega\hat{L}}\right)}\sqrt{\left(1 + \frac{\hat{G}}{j\omega\hat{C}}\right)} \approx j\omega\sqrt{\hat{L}\hat{C}}\left[1 - j\left(\frac{\hat{R}}{2\omega\hat{L}} + \frac{\hat{G}}{2\omega\hat{C}}\right)\right] \quad (9.87.1)$$

The attenuation constant can be approximated as

$$\alpha \approx \omega\sqrt{\hat{L}\hat{C}}\left(\frac{\hat{R}}{2\omega\hat{L}} + \frac{\hat{G}}{2\omega\hat{C}}\right) = \frac{1}{2}\sqrt{\hat{L}\hat{C}}\left(\frac{\hat{R}}{\hat{L}} + \frac{\hat{G}}{\hat{C}}\right) \quad (9.87.2)$$

The attenuation constant does not depend on frequency.



Lossy transmission lines (Ex)

Example 9.17: Find the complex propagation constant if the circuit elements satisfy the ratio $\hat{R}/\hat{L} = \hat{G}/\hat{C}$. Interpret this situation.

The complex propagation constant is:

$$\gamma = \alpha + j\beta = \sqrt{(\hat{R} + j\omega\hat{L})(\hat{G} + j\omega\hat{C})} = \sqrt{(\hat{R} + j\omega\hat{L})\left(\frac{\hat{R}\hat{C}}{\hat{L}} + j\omega\hat{C}\right)} = (\hat{R} + j\omega\hat{L})\sqrt{\frac{\hat{C}}{\hat{L}}}$$

The attenuation constant is independent on frequency, which implies no distortion of a signal as it propagates on this transmission line and a constant attenuation.

The characteristic impedance of this transmission line

$$Z_c = \sqrt{\frac{\hat{Z}}{\hat{Y}}} = \sqrt{\frac{\hat{R} + j\omega\hat{L}}{\hat{G} + j\omega\hat{C}}} = \sqrt{\frac{\hat{R} + j\omega\hat{L}}{(\hat{R}\hat{C}/\hat{L}) + j\omega\hat{C}}} = \sqrt{\frac{\hat{L}}{\hat{C}}}$$

does not depend on frequency. Therefore, this transmission line is distortionless.



Lossy transmission lines

Example 9.18: The attenuation on a 50Ω distortionless transmission line is 0.01 dB/m. The line has a capacitance of $0.1 \cdot 10^{-9}$ F/m. Find:

- Transmission line parameters: distributed inductance, resistance, conductance;
- The velocity of wave propagation.

Since the line is distortionless, the characteristic impedance is

$$Z_c = \sqrt{\frac{\hat{L}}{\hat{C}}} = 50 \Omega \Rightarrow \hat{L} = Z_c^2 \hat{C} = 50^2 \cdot 0.1 \cdot 10^{-9} = 2.5 \cdot 10^{-7} \text{ H/m}$$

The attenuation constant is:

$$\alpha = \hat{R} \sqrt{\frac{\hat{C}}{\hat{L}}} = 0.01 \text{ dB/m} = \{1 \text{ Np/m} = 20 \lg(e) = 8.686 \text{ dB/m}\} = \frac{0.01}{8.686} = 0.0012 \text{ Np/m}$$

Therefore:

$$\hat{R} = \alpha \sqrt{\frac{\hat{L}}{\hat{C}}} = \alpha Z_c = 0.0012 \cdot 50 = 0.0575 \Omega/m$$



Lossy transmission lines (Ex)

The distortionless line criterium is:

$$\frac{\hat{R}}{\hat{L}} = \frac{\hat{G}}{\hat{C}} \Rightarrow \hat{G} = \frac{\hat{R}\hat{C}}{\hat{L}} = \frac{\hat{R}}{Z_c^2} = \frac{0.0575}{50^2} = 2.3 \cdot 10^{-5} \text{ S/m}$$

The phase velocity is:

$$v = \frac{1}{\sqrt{\hat{L}\hat{C}}} = \frac{1}{\sqrt{2.5 \cdot 10^{-7} \cdot 0.1 \cdot 10^{-9}}} = 2 \cdot 10^8 \text{ m/s}$$



Dispersion and group velocity

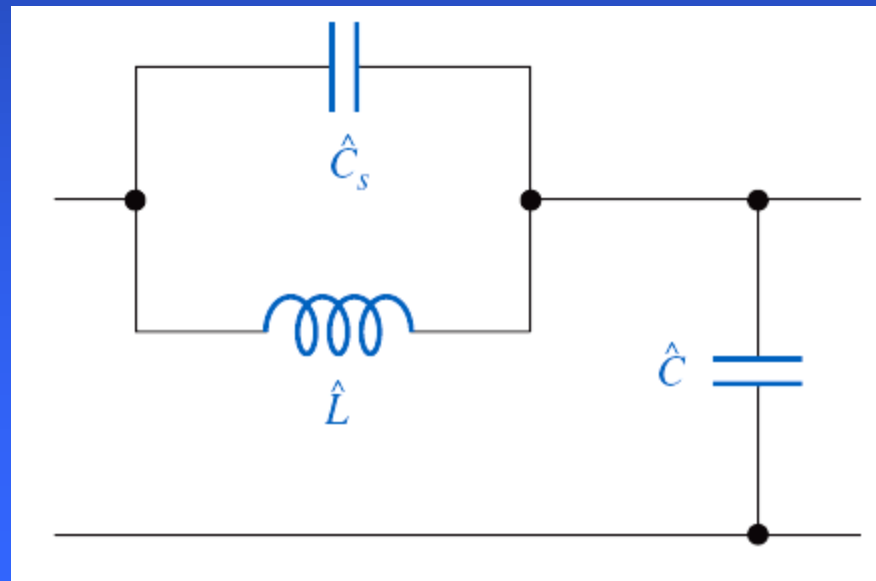
The losses of types considered previously (finite conductivity of conductors and nonzero conductivity of real dielectrics) lead to attenuation of wave amplitude as it propagates through the line.

When the wavelength is comparable with the physical dimensions of the line or when the permittivity of the dielectric depends on the frequency, another phenomenon called dispersion occurs.

We will model dispersion by the insertion of a distributed parasitic capacitance in parallel to the distributed inductance.

While developing the telegrapher's equation for this case, we note that the current entering the node will split into two portions:

$$I(z,t) = I_L(z,t) + I_C(z,t) \quad (9.91.1)$$



Dispersion and group velocity

The voltage drop across the capacitor is:

$$\frac{\partial V(z,t)}{\partial z} = -\hat{L} \frac{\partial I_L(z,t)}{\partial t} \quad (9.92.1)$$

The voltage drop across the parasitic (shunt) capacitor is:

$$\frac{\partial V(z,t)}{\partial z} = -\frac{1}{\hat{C}_s} \int I_c dt \quad (9.92.2)$$

Note: the units of this additional shunt capacitance are $F \cdot m$ rather than F/m .

The current passing through the shunt capacitor:

$$\frac{\partial I(z,t)}{\partial z} = -\hat{C} \frac{\partial V(z,t)}{\partial t} \quad (9.92.3)$$

The wave equation will be:

$$\frac{\partial^2 V(z,t)}{\partial z^2} - \hat{L}\hat{C} \frac{\partial^2 V(z,t)}{\partial t^2} + \hat{L}\hat{C}_s \frac{\partial^4 V(z,t)}{\partial z^2 \partial t^2} = 0 \quad (9.92.4)$$



Dispersion and group velocity

Assume that there is a time-harmonic signal generator connected to the infinitely long transmission line. The complex time-varying wave propagating through the line will be

$$V(z, t) = V_0 e^{j(\omega t - \beta z)} \quad (9.93.1)$$

Combining (9.92.4) and (9.93.1) leads to the dispersion relation (the terms in the square brackets) relating the propagation constant β to the frequency of the wave:

$$V_0 e^{j(\omega t - \beta z)} \left[(-j\beta)^2 - \hat{L}\hat{C}(j\omega)^2 + \hat{L}\hat{C}_s(j\omega)^2 (-j\beta)^2 \right] = 0 \quad (9.93.2)$$

Therefore, the propagation constant is a nonlinear function of frequency:

$$\beta = \pm \frac{\omega \sqrt{\hat{L}\hat{C}}}{\sqrt{1 - \omega^2 \hat{L}\hat{C}_s}} \quad (9.93.3)$$

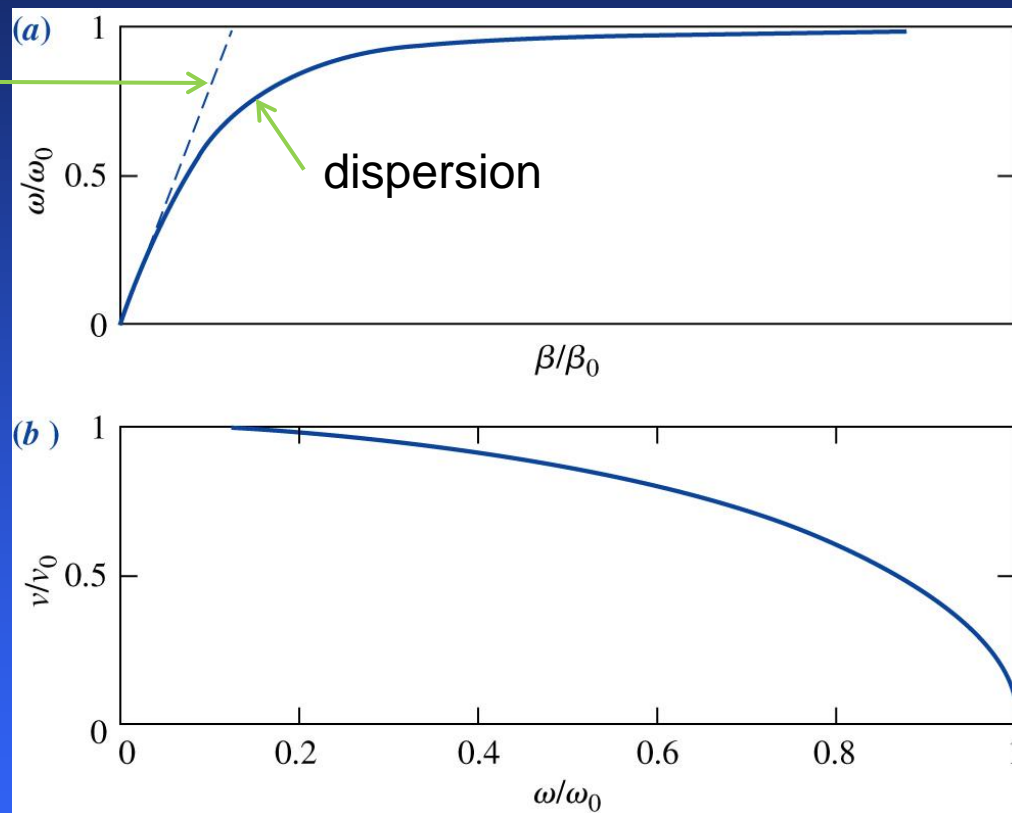


Dispersion and group velocity

no dispersion

The propagation constant depends on frequency – this phenomenon is called dispersion.

The phase velocity is a function of frequency also.



Dispersion and group velocity

We can consider dispersion as a low pass filter acting on a signal. The propagation constant will be a real number for frequencies less than the particular cutoff frequency ω_0 .

$$\omega_0 = \frac{1}{\sqrt{\hat{L}\hat{C}_s}} \quad (9.95.1)$$

This cutoff frequency is equal to the resonant frequency of the “tank” circuit. Above this frequency, the propagation constant will be imaginary, and the wave will not propagate.

In the non-dispersive frequency range (below the cutoff frequency), the velocity of propagation is

$$V_0 = \frac{1}{\sqrt{\hat{L}\hat{C}}} \quad (9.95.2)$$

The wave number below the cutoff frequency is:

$$\beta_0 = \frac{\omega_0}{v_0} \quad (9.95.3)$$

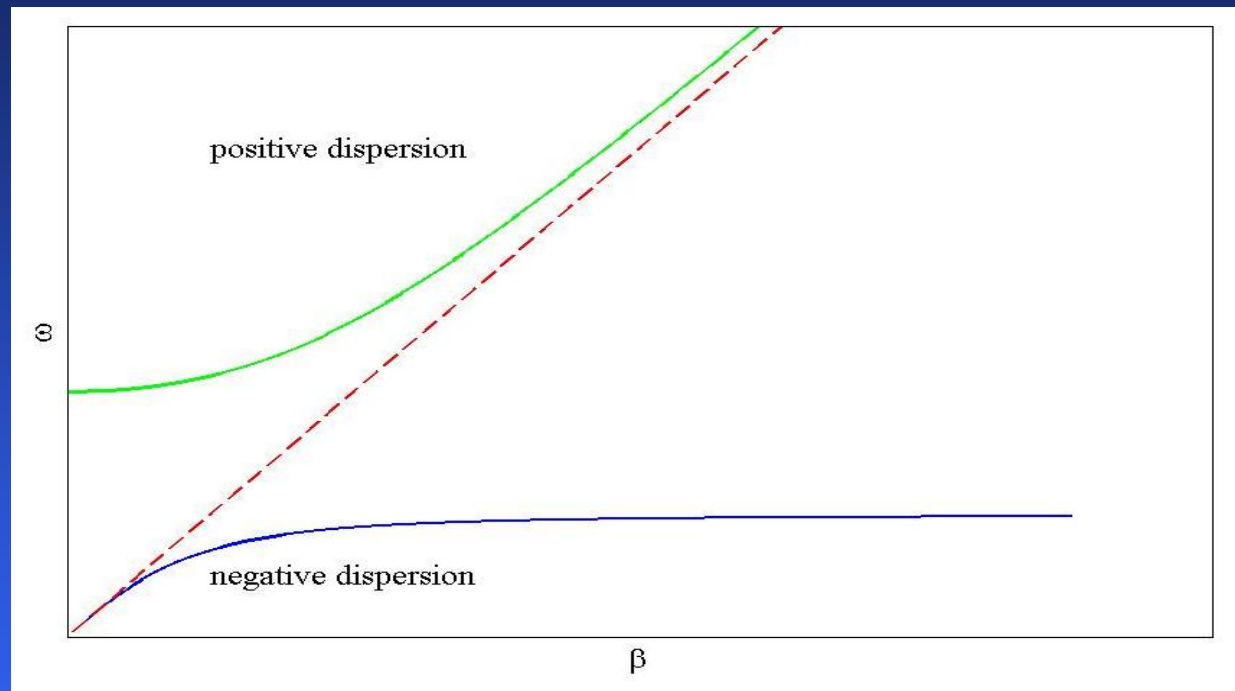


Dispersion and group velocity

Dispersion implies that the propagation constant depends on the frequency.

There are positive and negative dispersions.

Note that in the case of negative dispersion, waves of frequencies less than a certain cutoff frequency will propagate.



While for the positive dispersion, only the waves whose frequency exceeds the cutoff frequency will propagate.

Dispersion and group velocity

If two signals of different frequencies propagate through the same linear dispersive medium, we must employ the concept of group velocity...

if a narrow pulse propagates in a dispersive region, according to Fourier analysis, such a pulse consists of a number of high frequency components. Each of them will propagate with different phase velocity.

Let us assume that two waves of the same amplitude but slightly different frequencies propagate through the same dispersive medium. The frequencies are:

$$\omega_1 = \omega_0 + \Delta\omega; \quad \omega_2 = \omega_0 - \Delta\omega \quad (9.97.1)$$

The corresponding propagation constants are:

$$\beta_1 = \beta_0 + \Delta\beta; \quad \beta_2 = \beta_0 - \Delta\beta \quad (9.97.2)$$

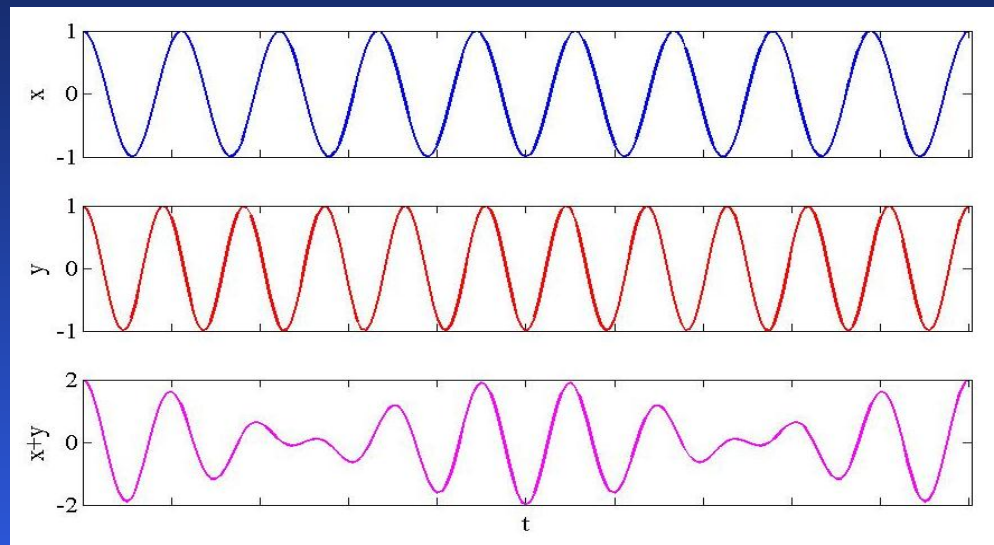
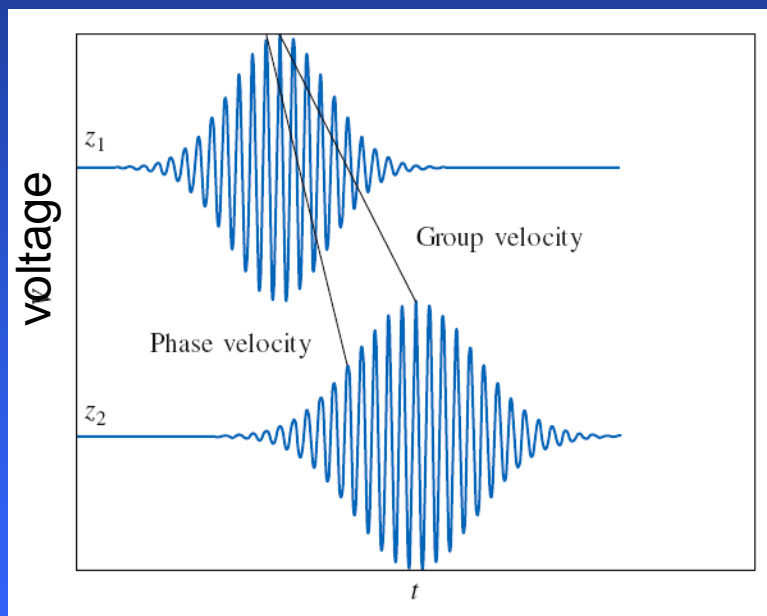
The total signal will be a sum of two waves:

$$\begin{aligned} V(z, t) &= V_0 \left[\cos(\omega_1 t - \beta_1 z) + \cos(\omega_2 t - \beta_2 z) \right] \\ &= 2V_0 \cos(\Delta\omega t - \Delta\beta z) \cos(\omega_0 t - \beta_0 z) \end{aligned} \quad (9.97.3)$$



Dispersion and group velocity

Summation of two time-harmonic waves of slightly different frequencies leads to constructive and destructive interference.



By detecting signals at two locations, we can track a point of constant phase propagating with the phase velocity:

$$v_p = \omega_0 / \beta_0 \quad (9.98.1)$$

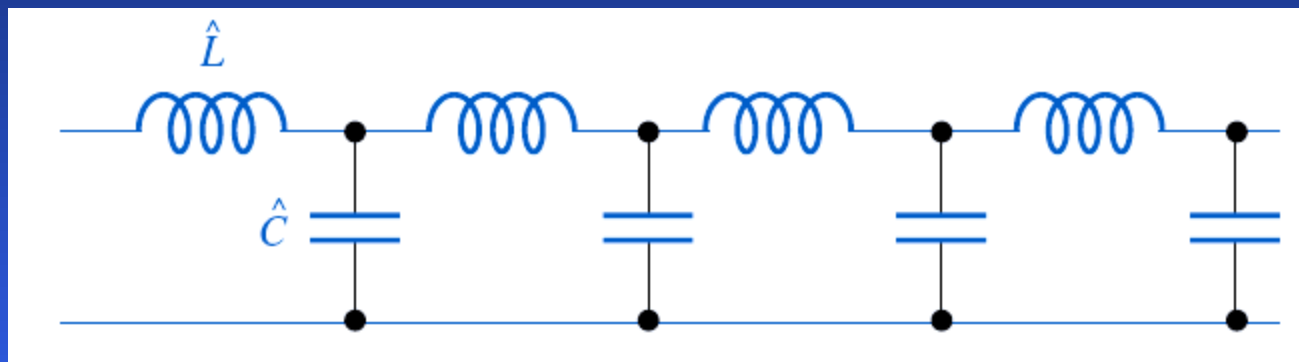
and the peak of the envelope propagating with the group velocity: $v_g = \Delta\omega / \Delta\beta$ (9.98.2)

In dispersive media, phase and group velocities can be considerably different!

Dispersion and group velocity (Ex)

Example 9.19: Find the phase and group velocities for:

- Normal transmission line;
- A line in which the elements are interchanged.



a) The propagation constant computed according to (9.87.1) will be:

$$\gamma = \alpha + j\beta = \sqrt{\hat{Y}\hat{Z}} = \sqrt{(j\omega\hat{L})(j\omega\hat{C})} = j\omega\sqrt{\hat{L}\hat{C}}$$

The phase velocity is

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\hat{L}\hat{C}}}$$

The group velocity is

$$v_g = \frac{\partial\omega}{\partial\beta} = 1 / \frac{\partial\beta}{\partial\omega} = \frac{1}{\sqrt{\hat{L}\hat{C}}}$$

Dispersion and group velocity (Ex)

The two velocities are equal in this case and both are independent on frequency.

b) The propagation constant computed according to (9.87.1) will be:

$$\gamma = \alpha + j\beta = \sqrt{\hat{Y}\hat{Z}} = \sqrt{\frac{1}{j\omega\hat{C}} \frac{1}{j\omega\hat{L}}} = -\frac{1}{j\omega\sqrt{\hat{L}\hat{C}}}$$

The phase velocity is

$$v_p = \frac{\omega}{\beta} = -\omega^2 \sqrt{\hat{L}\hat{C}}$$

The group velocity is

$$v_g = \frac{\partial\omega}{\partial\beta} = 1 / \frac{\partial\beta}{\partial\omega} = \omega^2 \sqrt{\hat{L}\hat{C}}$$

The phase and group velocities both depend on frequency and are in the opposite directions.

?? QUESTIONS ??

