13

VECTOR CALCULUS

13.2

Line Integrals

In this section, we will learn about:

Various aspects of line integrals

in planes, space, and vector fields.

In this section, we define an integral that is similar to a single integral except that, instead of integrating over an interval $[a, b]$, we integrate over a curve C.

- Such integrals are called line integrals.
- However, "curve integrals" would be better terminology.

LINE INTEGRALS

They were invented in the early 19th century to solve problems involving:

- Fluid flow
- Forces
- Electricity
- Magnetism

Equations 1

We start with a plane curve C given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad a \leq t \leq b$$

Equivalently, C can be given by the vector equation $r(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$.

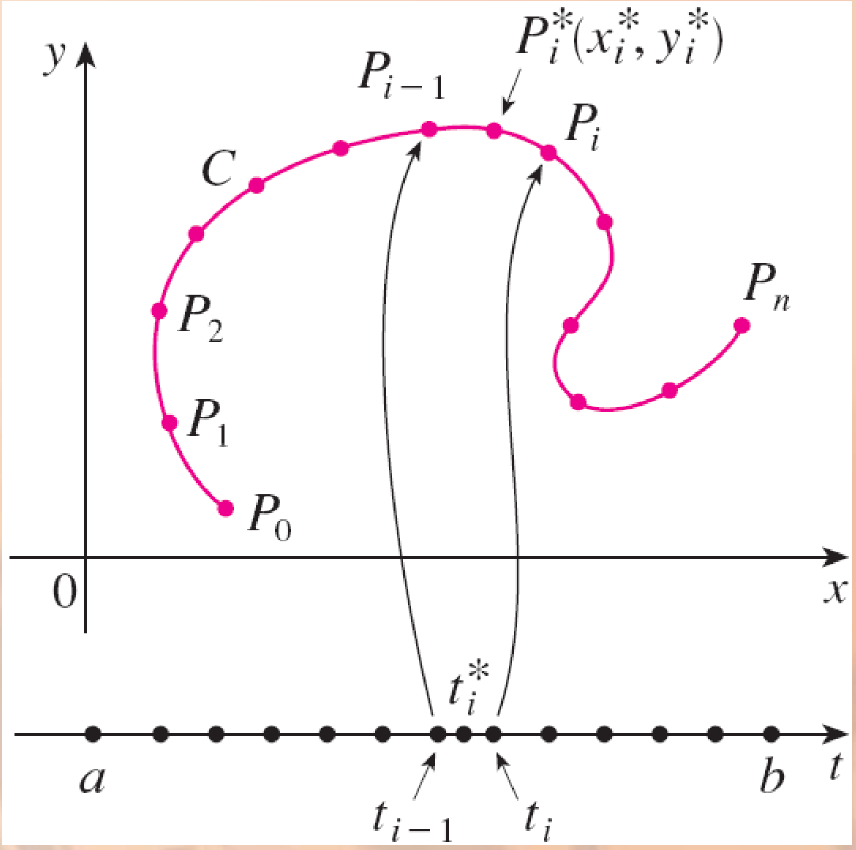
We assume that C is a smooth curve.

- This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$.

Let's divide the parameter interval [a, b] into n subintervals [t_{i-1}, t_i] of equal width.

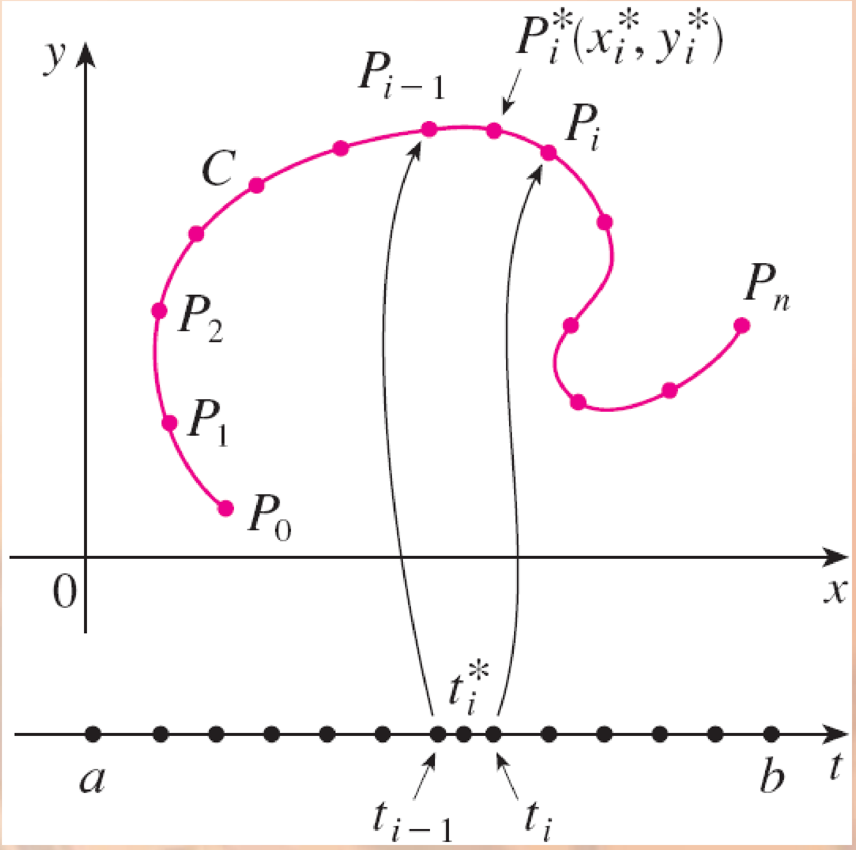
We let $x_i = x(t_i)$ and $y_i = y(t_i)$.

Then, the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$.



We choose any point $P_i^*(x_i^*, y_i^*)$ in the i th subarc.

- This corresponds to a point t_i^* in $[t_{i-1}, t_i]$.



Now, if *f* is any function of two variables whose domain includes the curve *C*, we:

1. Evaluate *f* at the point (x_i^*, y_i^*).

2. Multiply by the length Δs_i of the subarc.

3. Form the sum $\sum_{i=1}^{n} f\left(x_i^*, y_i^*\right)\Delta s_i$

   which is similar to a Riemann sum.

Then, we take the limit of these sums and make the following definition by analogy with a single integral.

Definition 2

If *f* is defined on a smooth curve *C* given by Equations 1, the line integral of *f* along *C* is:

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*, y_i^*\right)\Delta s_i$$

if this limit exists.

We found that the length of C is:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- A similar type of argument can be used to show that, if *f* is a continuous function, then the limit in Definition 2 always exists.

Then, this formula can be used to evaluate the line integral.

$$\int_C f(x, y)\,ds$$

$$= \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

The value of the line integral does not depend on the parametrization of the curve—provided the curve is traversed exactly once as t increases from a to b.

If $s(t)$ is the length of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

So, the way to remember Formula 3 is to express everything in terms of the parameter *t* :

- Use the parametric equations to express *x* and *y* in terms of *t* and write *ds* as:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

In the special case where *C* is the line segment that joins (*a*, 0) to (*b*, 0), using *x* as the parameter, we can write the parametric equations of *C* as:

$$x = x$$

$$y = 0$$

$$a \leq x \leq b$$

Formula 3 then becomes

$$\int_C f(x,y)\,ds = \int_a^b f(x,0)\,dx$$

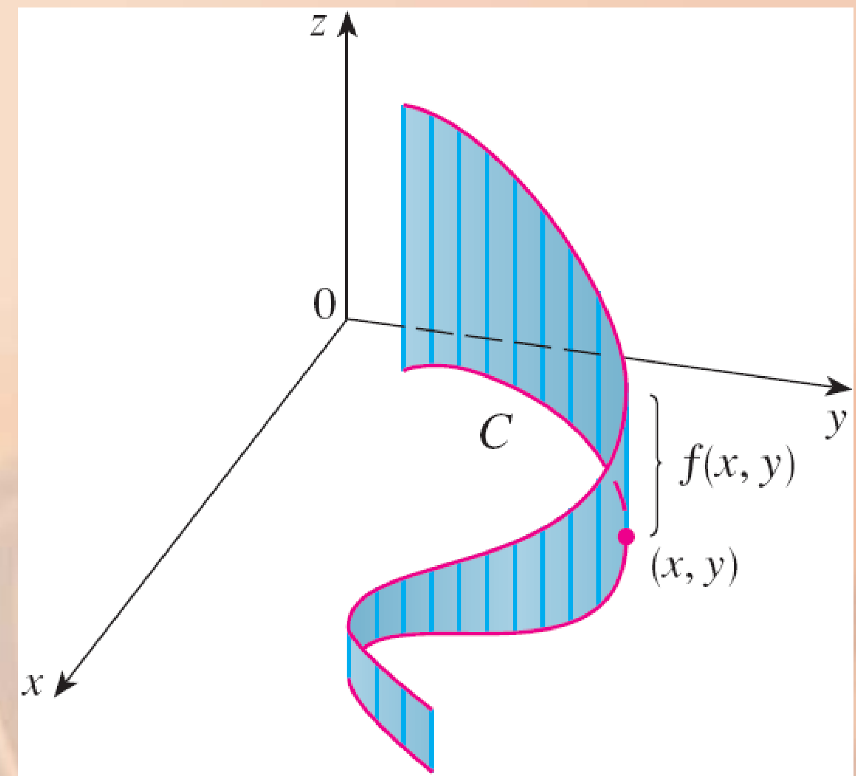
- So, the line integral reduces to an ordinary single integral in this case.

Just as for an ordinary single integral, we can interpret the line integral of a positive function as an area.

In fact, if $f(x, y) \geq 0$, $\int_C f(x, y)\,ds$ represents the area of one side of the "fence" or "curtain" shown here,

whose:

- Base is *C*.
- Height above the point (x, y) is $f(x, y)$.



Evaluate

$$\int_C \left(2 + x^2 y\right) ds$$

where C is the upper half of the unit circle $x^2 + y^2 = 1$

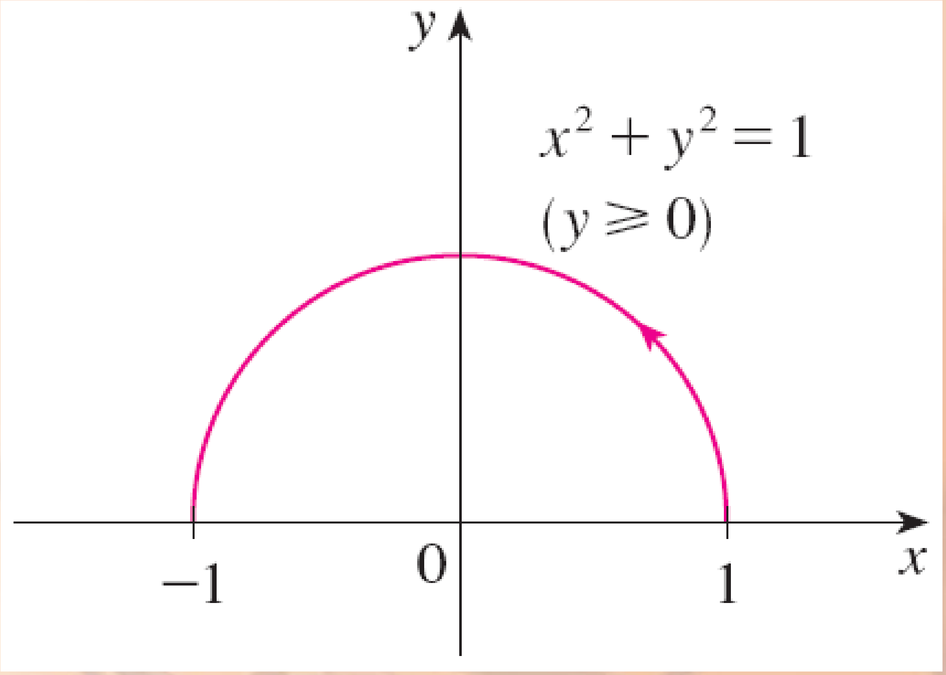
- To use Formula 3, we first need parametric equations to represent C.

Recall that the unit circle can be parametrized by means of the equations

$$x = \cos t \qquad y = \sin t$$

Also, the upper half of the circle is described by the parameter interval

$$0 \leq t \leq \pi$$



So, Formula 3 gives:

$$\int_C \left(2 + x^2 y\right) ds = \int_0^{\pi} \left(2 + \cos^2 t \sin t\right) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} \left(2 + \cos^2 t \sin t\right) \sqrt{\sin^2 t + \cos^2 t}\, dt$$

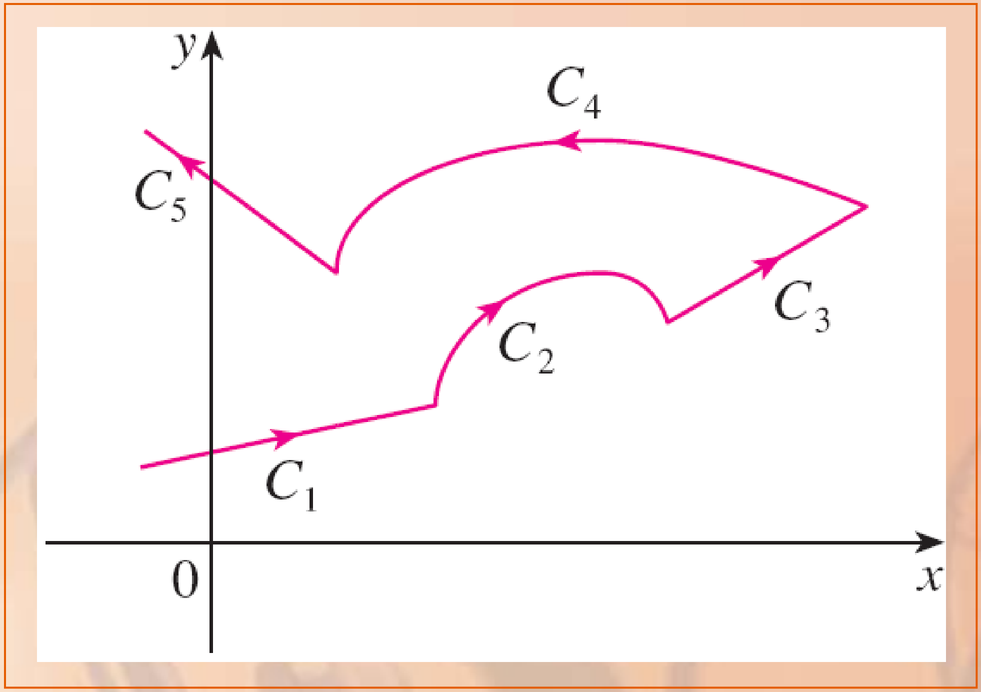
$$= \int_0^{\pi} \left(2 + \cos^2 t \sin t\right) dt$$

$$= \left[2t - \frac{\cos^3 t}{3}\right]_0^{\pi} = 2\pi + \tfrac{2}{3}$$

PIECEWISE-SMOOTH CURVE

Now, let C be a piecewise-smooth curve.

- That is, C is a union of a finite number of smooth curves C_1, C_2, …, C_n, where the initial point of C_{i+1} is the terminal point of C_i.



Then, we define the integral of *f* along *C* as the sum of the integrals of *f* along each of the smooth pieces of *C*:

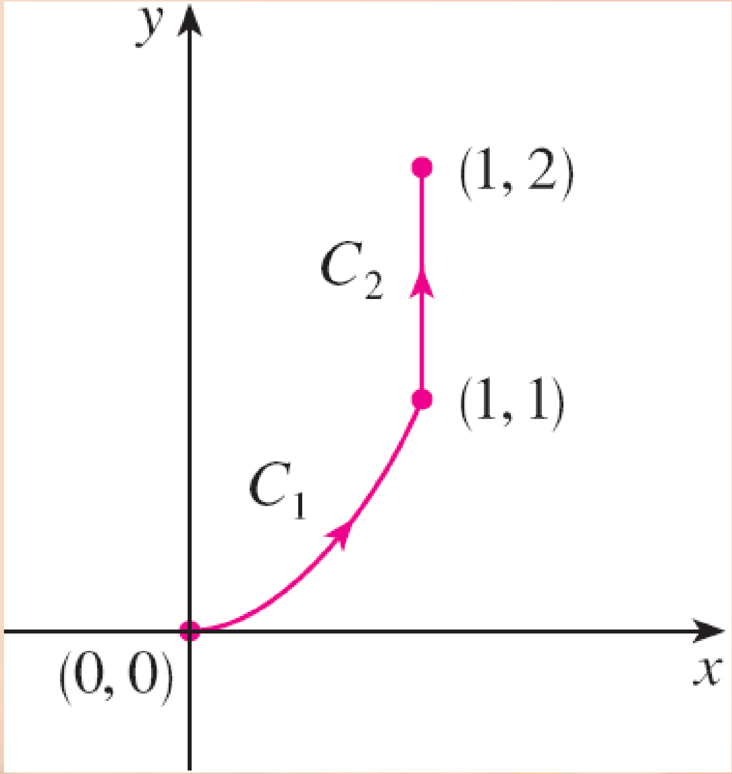
$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \ldots + \int_{C_n} f(x,y)\,ds$$

Evaluate

$$\int_C 2x\,ds$$

where C consists of the arc C_1 of the parabola $y = x^2$ from (0, 0) to (1, 1) followed by the vertical line segment C_2 from (1, 1) to (1, 2).

The curve is shown here.



C_1 is the graph of a function of *x*.

- So, we can choose *x* as the parameter.
- Then, the equations for C_1 become:

$$x = x \qquad y = x^2 \qquad 0 \leq x \leq 1$$

Therefore,

$$\int_{C_1} 2x\,ds = \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^1 2x\sqrt{1+4x^2}\,dx$$

$$= \tfrac{1}{4}\cdot\tfrac{2}{3}\left(1+4x^2\right)^{3/2}\Big]_0^1$$

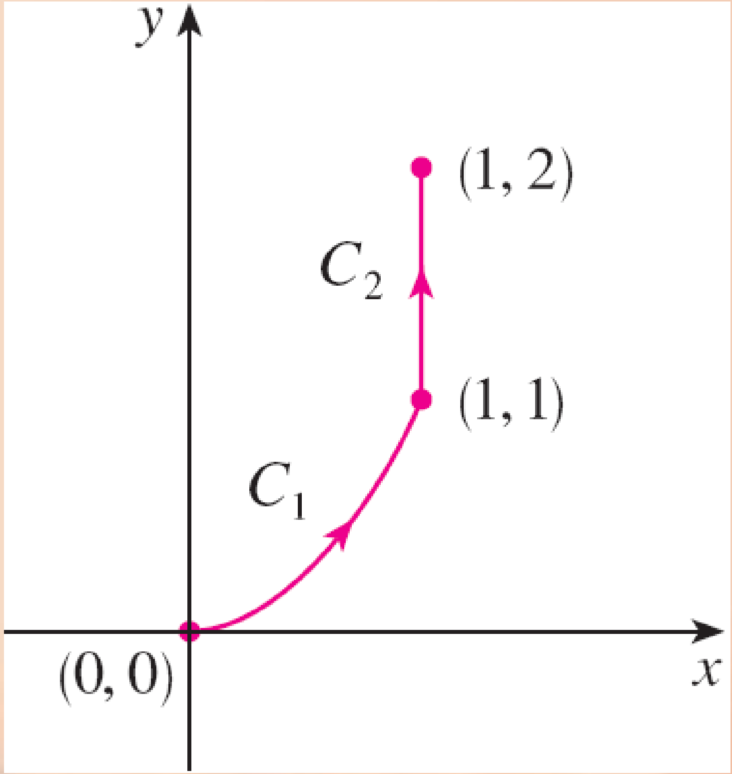
$$= \frac{5\sqrt{5}-1}{6}$$

Example 2

On C_2, we choose y as the parameter.

- So, the equations of C_2 are:
  $x = 1 \qquad y = 1 \qquad 1 \leq y \leq 2$
  and

$$\int_{C_2} 2x\,ds = \int_1^2 2(1)\sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\,dy = \int_1^2 2\,dy = 2$$



Thus,

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds$$

$$= \frac{5\sqrt{5}-1}{6} + 2$$

LINE INTEGRALS

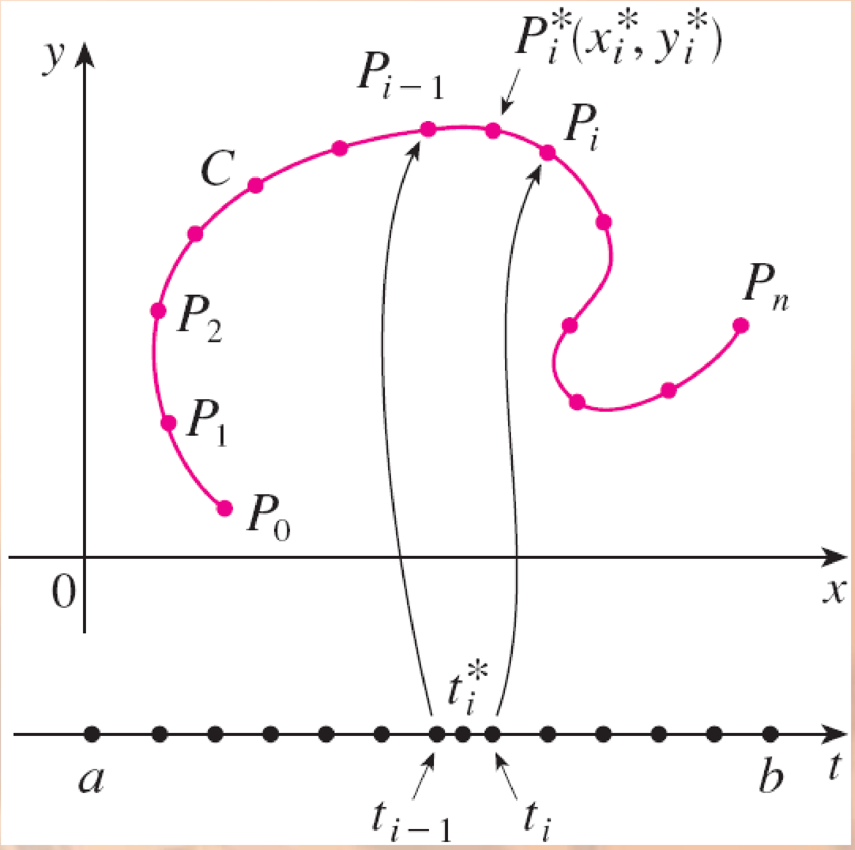
Any physical interpretation of a line integral

$$\int_C f(x, y)\,ds$$

depends on the physical interpretation of the function *f*.

- Suppose that $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve *C*.

Then, the mass of the part of the wire from $P_{i\text{-}1}$ to P_i in this figure is approximately $\rho(x_i^*, y_i^*)\ \Delta s_i$.



- So, the total mass of the wire is approximately $\sum \rho(x_i^*, y_i^*)\ \Delta s_i$.

MASS

By taking more and more points on the curve, we obtain the mass *m* of the wire as the limiting value of these approximations:

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho\left(x_i^*, y_i^*\right) \Delta s_i$$

$$= \int_C \rho(x, y)\, ds$$

MASS

For example, if $f(x, y) = 2 + x^2y$ represents the density of a semicircular wire, then the integral in Example 1 would represent the mass of the wire.

The center of mass of the wire with density function ρ is located at the point $\left(\bar{x}, \bar{y}\right)$, where:

$$\bar{x} = \frac{1}{m}\int_C x\rho(x, y)\, ds$$

$$\bar{y} = \frac{1}{m}\int_C y\rho(x, y)\, ds$$

A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geq 0$, and is thicker near its base than near the top.

- Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

As in Example 1, we use the parametrization

$$x = \cos t \qquad y = \sin t \qquad 0 \le t \le \pi$$

and find that $ds = dt$.

The linear density is $\rho(x, y) = k(1 - y)$

where k is a constant.

So, the mass of the wire is:

$$m = \int_C k(1 - y)\,ds = \int_0^{\pi} k(1 - \sin t)\,dt$$
$$= k\left[t + \cos t\right]_0^{\pi}$$
$$= k(\pi - 2)$$

From Equations 4, we have:

$$\bar{y} = \frac{1}{m}\int_C y\,\rho(x,y)\,ds = \frac{1}{k(\pi-2)}\int_C y\,k(1-y)\,ds$$

$$= \frac{1}{\pi-2}\int_0^{\pi}\left(\sin t - \sin^2 t\right)dt$$

$$= \frac{1}{\pi-2}\left[-\cos t - \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right]_0^{\pi}$$

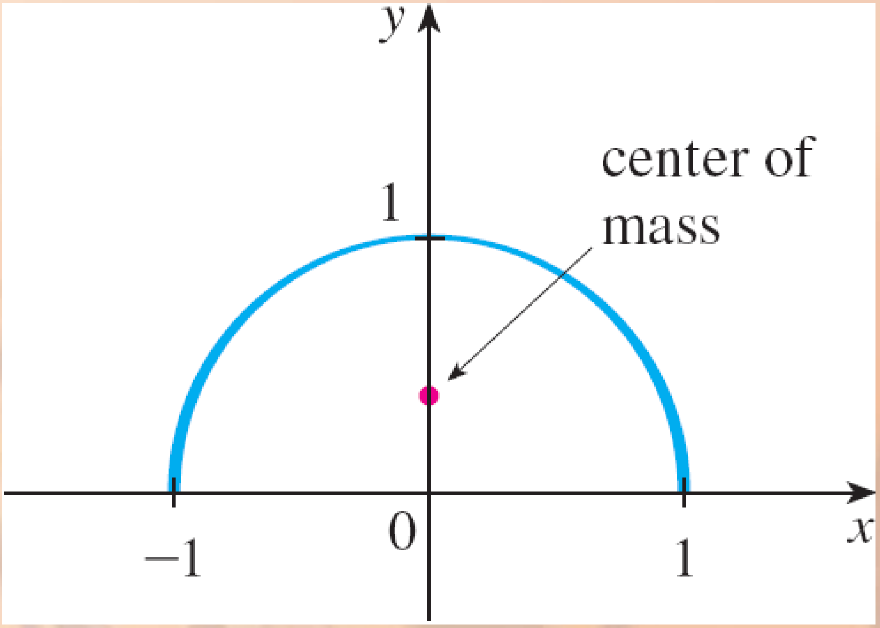
$$= \frac{4-\pi}{2(\pi-2)}$$

By symmetry, we see that $\bar{x} = 0$.

So, the center of mass is:

$$\left(0, \frac{4-\pi}{2(\pi-2)}\right)$$

$$\approx (0, 0.38)$$



Two other line integrals are obtained by replacing Δs_i, in Definition 2, by either:

- $\Delta x_i = x_i - x_{i\text{-}1}$
- $\Delta y_i = y_i - y_{i\text{-}1}$

They are called the line integrals of *f* along *C* with respect to *x* and *y*:

$$\int_C f(x, y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*, y_i^*\right) \Delta x_i$$

$$\int_C f(x, y)\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*, y_i^*\right) \Delta y_i$$

ARC LENGTH

When we want to distinguish the original line integral $\int_C f(x, y)\, ds$ from those in Equations 5 and 6, we call it the line integral with respect to arc length.

TERMS OF *t*

The following formulas say that line integrals with respect to *x* and *y* can also be evaluated by expressing everything in terms of *t*:

$$x = x(t)$$

$$y = y(t)$$

$$dx = x'(t)\, dt$$

$$dy = y'(t)\, dt$$

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

ABBREVIATING

It frequently happens that line integrals with respect to *x* and *y* occur together.

- When this happens, it's customary to abbreviate by writing

$$\int_C P(x, y)\,dx + \int_C Q(x, y)\,dy$$

$$= \int_C P(x, y)\,dx + Q(x, y)\,dy$$

LINE INTEGRALS

When we are setting up a line integral, sometimes, the most difficult thing is to think of a parametric representation for a curve whose geometric description is given.

- In particular, we often need to parametrize a line segment.

So, it's useful to remember that a vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by:

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\,\mathbf{r}_1 \qquad 0 \le t \le 1$$

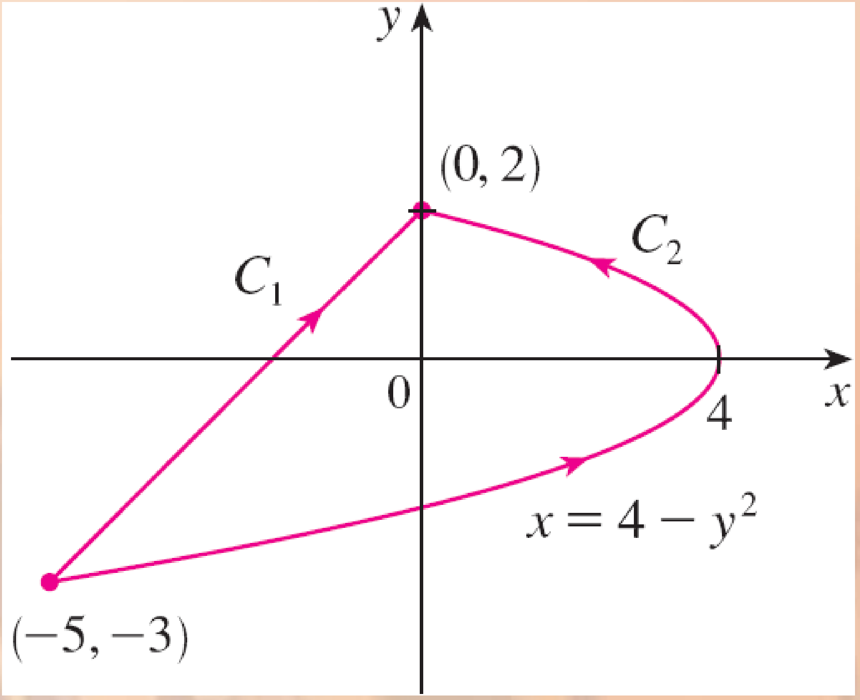
Example 4

Evaluate $\int_C y^2\,dx + x\,dy$

where

a. $C = C_1$ is the line segment from (–5, –3) to (0, 2)

b. $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from (–5, –3) to (0, 2).



A parametric representation for the line segment is:

$$x = 5t - 5 \qquad y = 5t - 3 \qquad 0 \leq t \leq 1$$

- Use Equation 8 with $\mathbf{r}_0 = \langle -5, 3 \rangle$ and $\mathbf{r}_1 = \langle 0, 2 \rangle$.

Then, *dx* = 5 *dt*, *dy* = 5 *dt*,

and Formulas 7 give:

$$\int_{C_1} y^2 dx + x\, dy = \int_0^1 (5t-3)^2 (5\,dt) + (5t-5)(5\,dt)$$

$$= 5\int_0^1 \left(25t^2 - 25t + 4\right) dt$$

$$= 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right]_0^1 = -\frac{5}{6}$$

The parabola is given as a function of *y*.

So, let's take *y* as the parameter and write C_2 as:

$$x = 4 - y^2 \qquad y = y \qquad -3 \le y \le 2$$

Then, $dx = -2y\,dy$

and, by Formulas 7, we have:

$$\int_{C_2} y^2 dx + x\,dy = \int_{-3}^{2} y^2(-2y)\,dy + (4 - y^2)\,dy$$

$$= \int_{-3}^{2} (-2y^3 - y^2 + 4)\,dy$$

$$= \left[-\frac{y^4}{2} - \frac{y^3}{3} + 4y \right]_{-3}^{2} = 40\tfrac{5}{6}$$

Notice that we got different answers in parts a and b of Example 4 although the two curves had the same endpoints.

- Thus, in general, the value of a line integral depends not just on the endpoints of the curve but also on the path.
- However, see Section 13.3 for conditions under which it is independent of the path.

Notice also that the answers in Example 4 depend on the direction, or orientation, of the curve.

- If $-C_1$ denotes the line segment from (0, 2) to (–5, –3), you can verify, using the parametrization

$$x = -5t \qquad y = 2 - 5t \qquad 0 \le t \le 1$$

that

$$\int_{-C_1} y^2\,dx + x\,dy = \tfrac{5}{6}$$

CURVE ORIENTATION

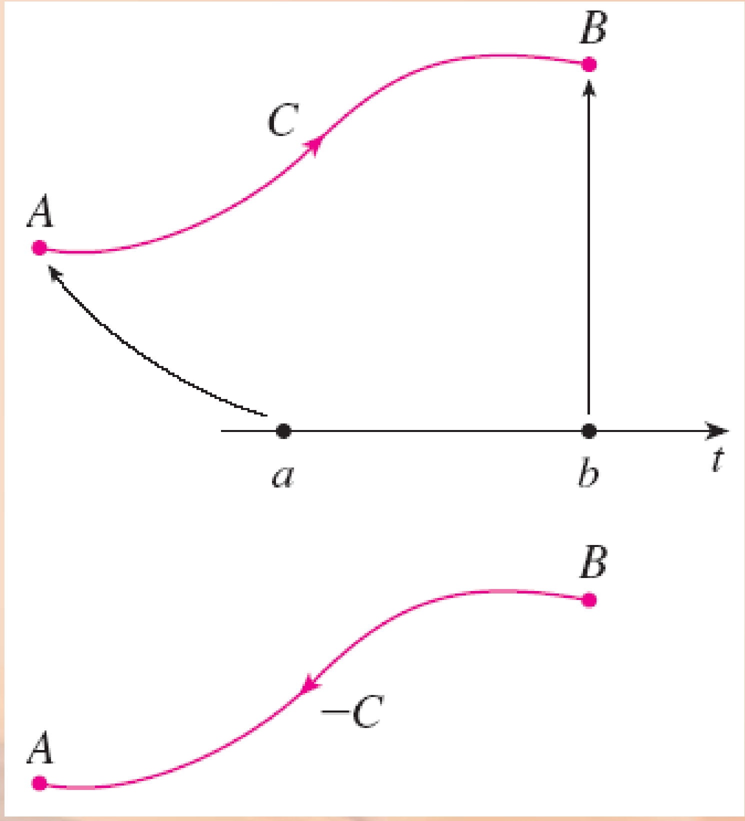
In general, a given parametrization

$$x = x(t),\ y = y(t),\ a \le t \le b$$

determines an orientation of a curve *C*, with the positive direction corresponding to increasing values of the parameter *t*.

For instance, here

- The initial point *A* corresponds to the parameter value.
- The terminal point *B* corresponds to $t = b$.



CURVE ORIENTATION

If –*C* denotes the curve consisting of the same points as *C* but with the opposite orientation (from initial point *B* to terminal point *A* in the previous figure), we have:

$$\int_{-C} f(x, y)\,dx = -\int_{C} f(x, y)\,dx$$

$$\int_{-C} f(x, y)\,dy = -\int_{C} f(x, y)\,dy$$

CURVE ORIENTATION

However, if we integrate with respect to arc length, the value of the line integral does not change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y)\, ds = \int_{C} f(x, y)\, ds$$

- This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of C.

LINE INTEGRALS IN SPACE

We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

or by a vector equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

LINE INTEGRALS IN SPACE

Suppose *f* is a function of three variables that is continuous on some region containing *C*.

- Then, we define the line integral of *f* along *C* (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*, y_i^*, z_i^*\right)\Delta s_i$$

We evaluate it using a formula similar to Formula 3:

$$\int_C f(x, y, z)\, ds$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Observe that the integrals in both Formulas 3 and 9 can be written in the more compact vector notation

$$\int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt$$

For the special case $f(x, y, z) = 1$, we get:

$$\int_C ds = \int_a^b |\mathbf{r}'(t)|\,dt = L$$

where L is the length of the curve C.

LINE INTEGRALS IN SPACE

Line integrals along *C* with respect to *x*, *y*, and *z* can also be defined.

- For example,

$$\int_C f(x,y,z)\,dz = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*, y_i^*, z_i^*\right)\Delta z_i$$

$$= \int_a^b f(x(t), y(t), z(t))\, z'(t)\, dt$$

Formula 10

Thus, as with line integrals in the plane, we evaluate integrals of the form

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

by expressing everything (*x*, *y*, *z*, *dx*, *dy*, *dz*) in terms of the parameter *t*.

 Example 5

Evaluate

$$\int_C y \sin z \, ds$$

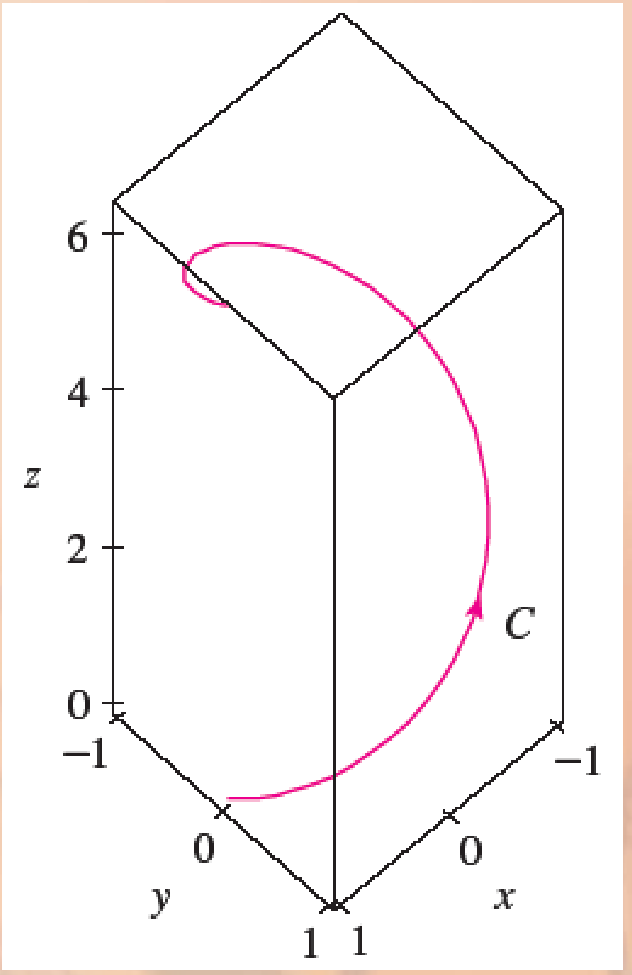
where C is the circular helix given by the equations

$$x = \cos t$$

$$y = \sin t$$

$$z = t$$

$0 \leq t \leq 2\pi$



Formula 9 gives:

$$\int_C y \sin z \, ds$$

$$= \int_0^{2\pi} (\sin t) \sin t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

$$= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1} \, dt$$

$$= \sqrt{2} \int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2t) \, dt$$

$$= \frac{\sqrt{2}}{2} \left[t - \tfrac{1}{2} \sin 2t\right]_0^{2\pi} = \sqrt{2}\pi$$

Example 6

Evaluate

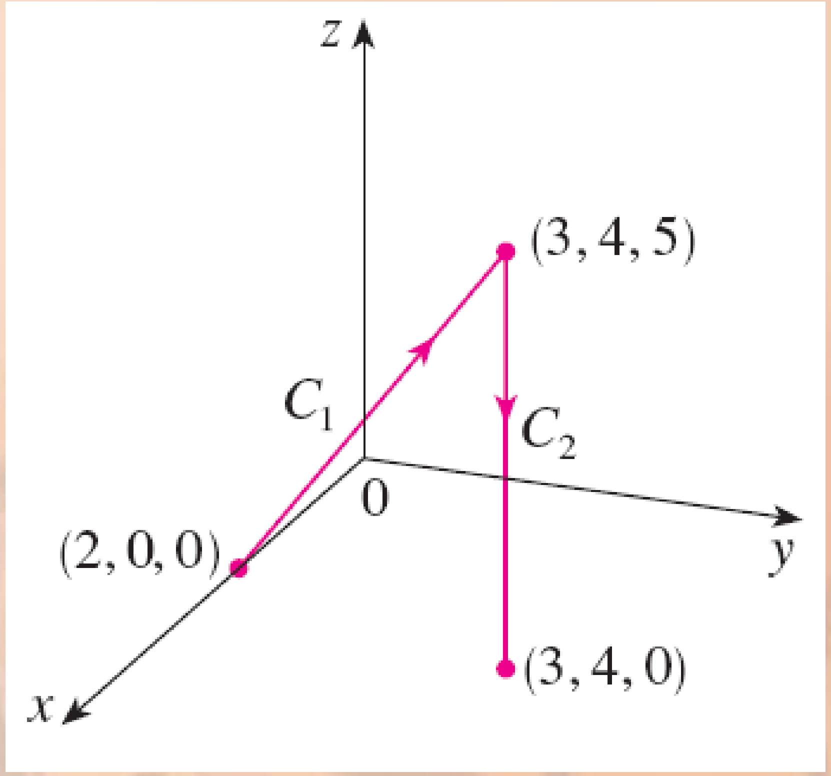
$$\int_C y\,dx + z\,dy + x\,dz$$

where C consists of the line segment C_1 from (2, 0, 0) to (3, 4, 5), followed by the vertical line segment C_2 from (3, 4, 5) to (3, 4, 0).

The curve C is shown.

- Using Equation 8, we write C_1 as:

  $$r(t) = (1 - t)\langle 2, 0, 0\rangle + t\langle 3, 4, 5\rangle = \langle 2 + t, 4t, 5t\rangle$$

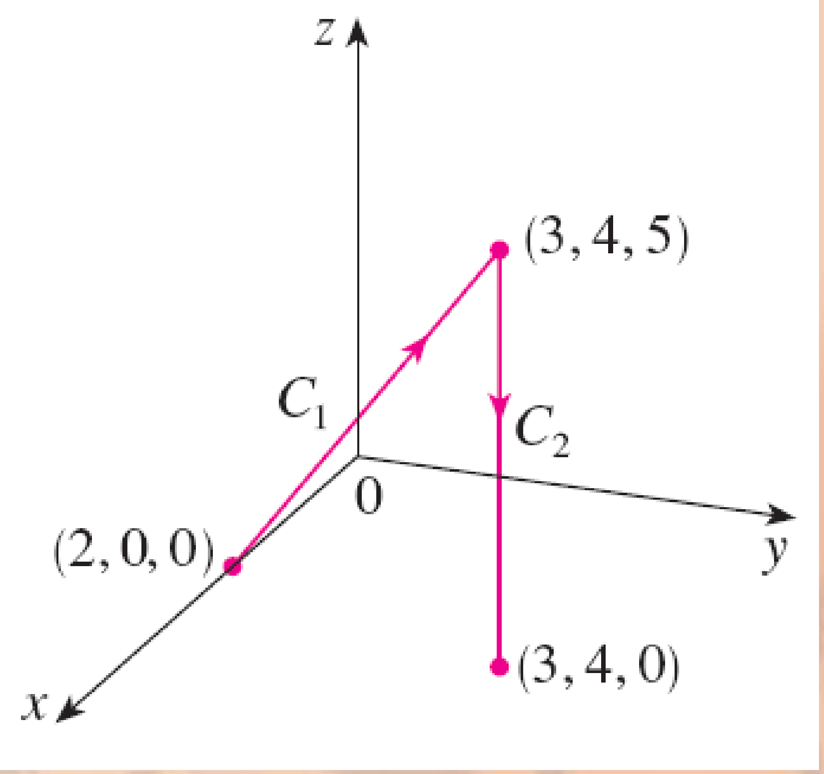


LINE INTEGRALS IN SPACE

- Alternatively, in parametric form, we write C_1 as:

$$x = 2 + t$$
$$y = 4t$$
$$z = 5t$$

$$0 \le t \le 1$$



- Thus,

$$\int_{C_1} y\,dx + z\,dy + x\,dz$$

$$= \int_0^1 (4t)\,dt + (5t)\,4\,dt + (2+t)\,5\,dt$$

$$= \int_0^1 (10 + 29t)\,dt$$

$$= 10t + 29\frac{t^2}{2}\Big]_0^1 = 24.5$$

Likewise, C_2 can be written in the form

$$r(t) = (1 - t)\langle 3, 4, 5\rangle + t\langle 3, 4, 0\rangle$$

$$= \langle 3, 4, 5 - 5t\rangle$$

or

$$x = 3 \qquad y = 4 \qquad z = 5 - 5t \qquad 0 \leq t \leq 1$$

Then, *dx* = 0 = *dy*.

So, $$\int_{C_1} y\,dx + z\,dy + x\,dz = \int_3^1 3(-5)\,dt$$

$$= -15$$

- Adding the values of these integrals, we obtain:

$$\int_{C_1} y\,dx + z\,dy + x\,dz = 24.5 - 15$$

$$= 9.5$$

Recall that the work done

by a variable force *f*(*x*) in moving a particle

from *a* to *b* along the *x*-axis is:

$$W = \int_a^b f(x)\,dx$$

We found that the work done by a constant force **F** in moving an object from a point *P* to another point in space is:

$$W = \mathbf{F} \cdot \mathbf{D}$$

where **D** = $\overrightarrow{PQ}$ is the displacement vector.

LINE INTEGRALS OF VECTOR FIELDS

Now, suppose that

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

is a continuous force field on ° ^{3},

such as:

- The gravitational field
- The electric force field

LINE INTEGRALS OF VECTOR FIELDS

A force field on °^{3} could be regarded as a special case where $R = 0$ and P and Q depend only on x and y.

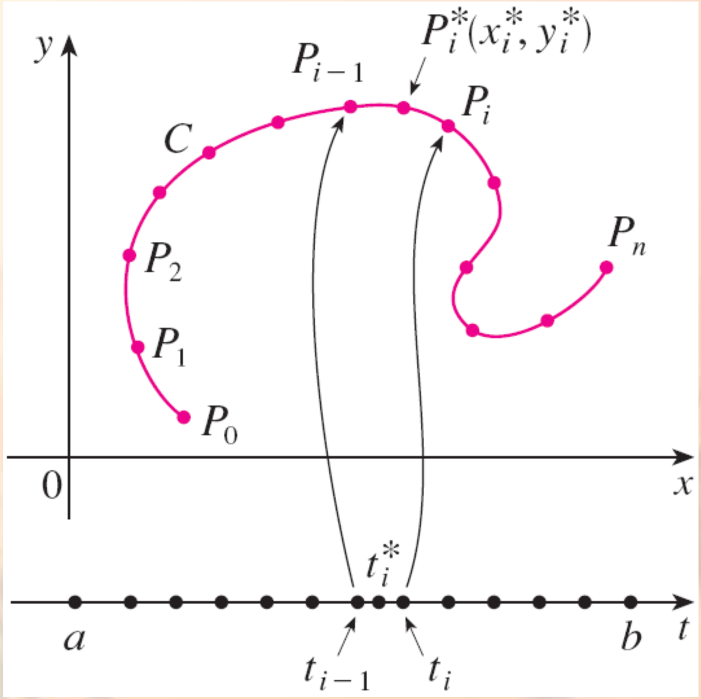
- We wish to compute the work done by this force in moving a particle along a smooth curve C.

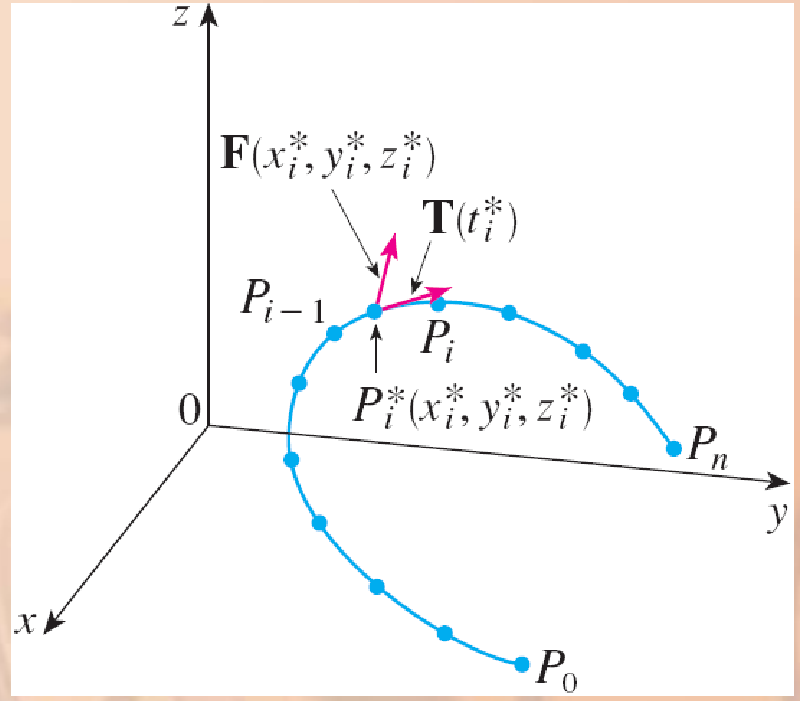
We divide C into subarcs $P_{i-1}P_i$ with lengths Δs_i by dividing the parameter interval $[a, b]$ into subintervals of equal width.

LINE INTEGRALS OF VECTOR FIELDS

The first figure shows the two-dimensional case.

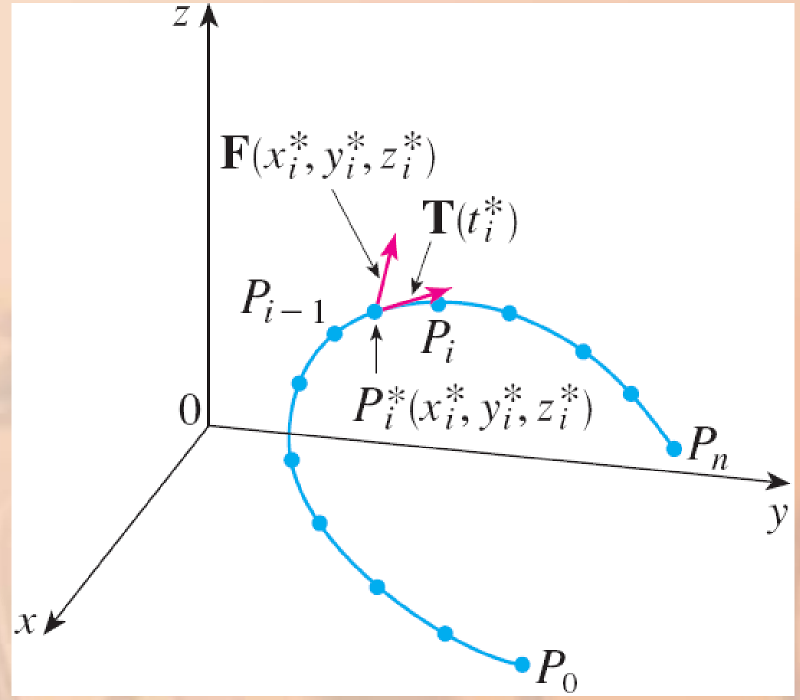
The second shows the three-dimensional one.



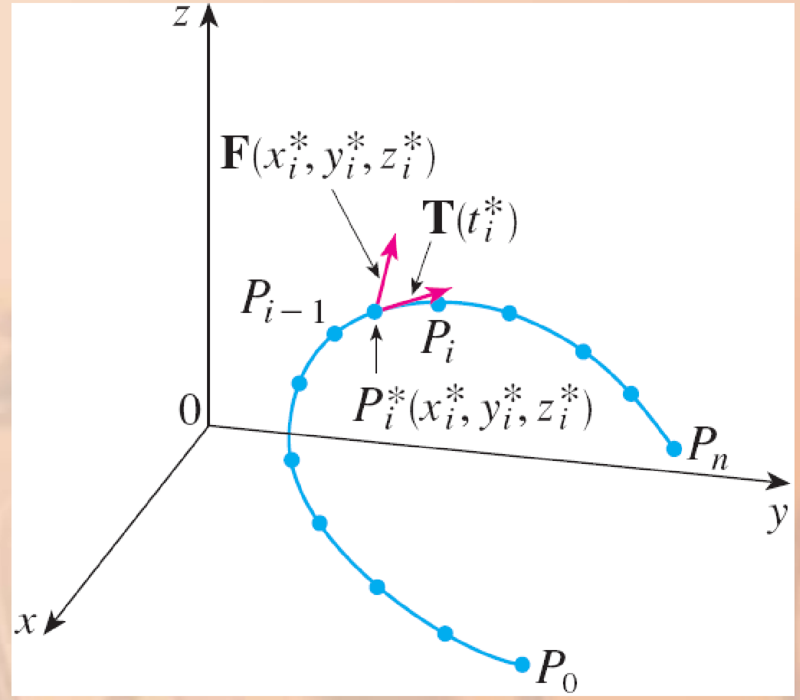


LINE INTEGRALS OF VECTOR FIELDS

Choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$ on the *i* th subarc corresponding to the parameter value t_i^*.



If Δs_i is small, then as the particle moves from $P_{i\text{-}1}$ to P_i along the curve, it proceeds approximately in the direction of $\mathbf{T}(t_i^*)$, the unit tangent vector at P_i^*.



Thus, the work done by the force **F** in moving the particle P_{i-1} from to P_i is approximately

$$\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \, \mathbf{T}(t_i^*)]$$
$$= [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)] \, \Delta s_i$$

The total work done in moving the particle along *C* is approximately

$$\sum_{i=1}^{n} \left[\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*) \right] \Delta s_i$$

where **T**(*x*, *y*, *z*) is the unit tangent vector at the point (*x*, *y*, *z*) on *C*.

Intuitively, we see that these approximations ought to become better as *n* becomes larger.

Thus, we define the work *W* done by the force field **F** as the limit of the Riemann sums in Formula 11, namely,

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\, ds = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

- This says that work is the line integral with respect to arc length of the tangential component of the force.

If the curve C is given by the vector equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

then

$$\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$$

So, using Equation 9, we can rewrite Equation 12 in the form

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)|\, dt$$

$$= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

This integral is often abbreviated as

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

and occurs in other areas of physics as well.

- Thus, we make the following definition for the line integral of any continuous vector field.

Definition 13

Let **F** be a continuous vector field defined on a smooth curve C given by a vector function **r**(t), $a \leq t \leq b$.

Then, the line integral of **F** along C is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

When using Definition 13, remember $\mathbf{F}(\mathbf{r}(t))$ is just an abbreviation for

$$\mathbf{F}(x(t), y(t), z(t))$$

- So, we evaluate $\mathbf{F}(\mathbf{r}(t))$ simply by putting
  $$x = x(t),\ y = y(t), \text{ and } z = z(t)$$
  in the expression for $\mathbf{F}(x, y, z)$.

- Notice also that we can formally write $d\mathbf{r} = \mathbf{r}'(t)\, dt$.

Find the work done by the force field

$$\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$$

in moving a particle along

the quarter-circle

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \quad 0 \le t \le \pi/2$$

Since $x = \cos t$ and $y = \sin t$,

we have:

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\ \mathbf{i} - \cos t \sin t\ \mathbf{j}$$

and

$$\mathbf{r}'(t) = -\sin t\ \mathbf{i} + \cos t\ \mathbf{j}$$

Therefore, the work done is:

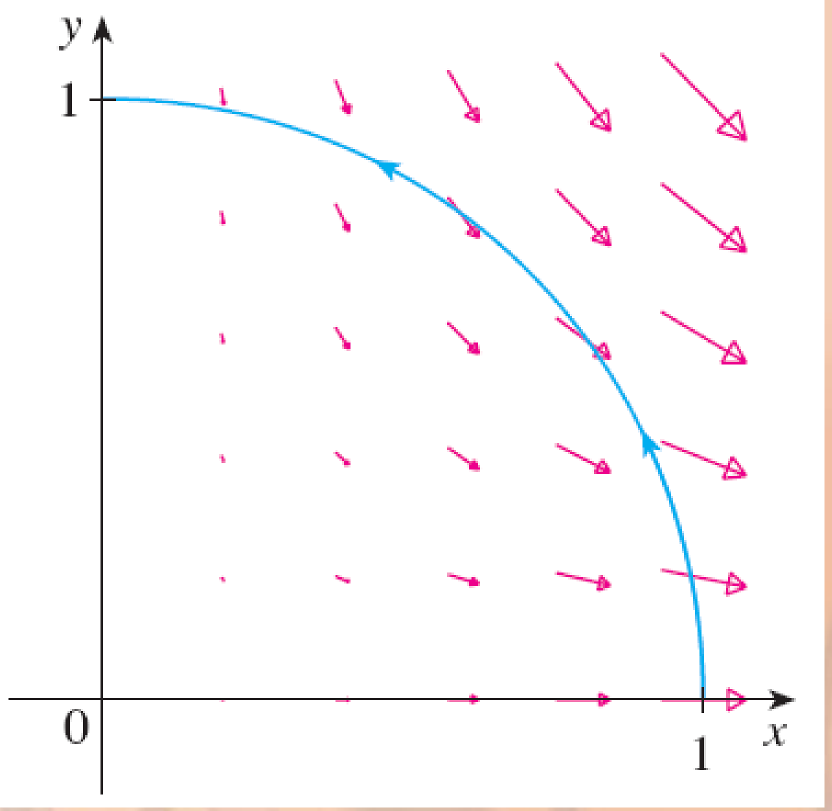
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{\pi/2} \left(-2\cos^2 t \sin t\right) dt$$

$$= 2\frac{\cos^3 t}{3}\Big]_0^{\pi/2} = -\frac{2}{3}$$

The figure shows the force field and the curve in Example 7.

- The work done is negative because the field impedes movement along the curve.



Note

Although $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ and integrals with respect to arc length are unchanged when orientation is reversed, it is still true that:

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

- This is because the unit tangent vector **T** is replaced by its negative when *C* is replaced by –*C*.

Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where:

- $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$
- C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leq t \leq 1$$

We have:

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = t^3\,\mathbf{i} + t^5\,\mathbf{j} + t^4\,\mathbf{k}$$

Thus,

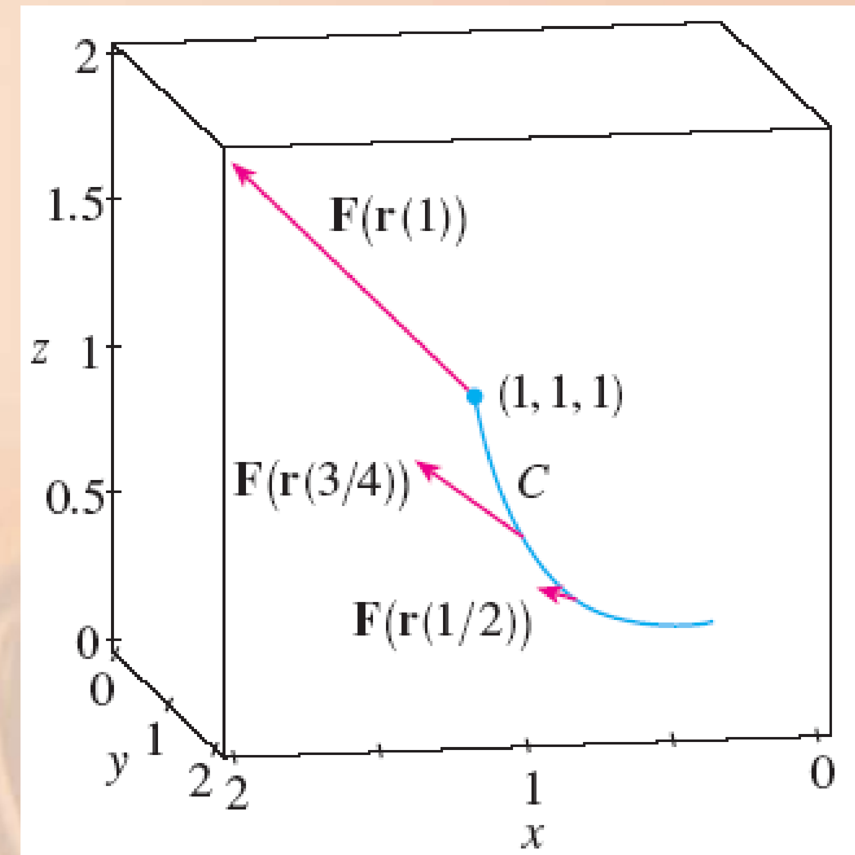
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t) \cdot \mathbf{r}'(t))\, dt$$

$$= \int_0^1 \left(t^3 + 5t^6\right) dt$$

$$= \frac{t^4}{4} + \frac{5t^7}{7}\bigg]_0^1 = \frac{27}{28}$$

VECTOR FIELDS

The figure shows the twisted cubic in Example 8 and some typical vectors acting at three points on *C*.



Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields.

Suppose the vector field **F** on $\mathbb{R}^3$ is given in component form by:

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

- We use Definition 13 to compute its line integral along *C*, as follows.

VECTOR & SCALAR FIELDS

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_a^b (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k})\, dt$$

$$= \int_a^b \begin{bmatrix} P(x(t), y(t), z(t))\, x'(t) \\ +Q(x(t), y(t), z(t))\, y'(t) \\ +R(x(t), y(t), z(t))\, z'(t) \end{bmatrix} dt$$

However, that last integral is precisely the line integral in Formula 10.

Hence, we have:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$$

where $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$

For example, the integral

$$\int_C y\,dx + z\,dy + x\,dz$$

in Example 6 could be expressed as

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where

$$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$$