13

VECTOR CALCULUS

13.3

Fundamental Theorem for Line Integrals

In this section, we will learn about:

The Fundamental Theorem for line integrals and determining conservative vector fields.

Recall from

the Fundamental Theorem of Calculus

(FTC2) can be written as:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

where F' is continuous on $[a, b]$.

We also called Equation 1 the Net Change Theorem:

- The integral of a rate of change is the net change.

FUNDAMENTAL THEOREM (FT) FOR LINE INTEGRALS

Suppose we think of the gradient vector ∇f of a function *f* of two or three variables as a sort of derivative of *f*.

Then, the following theorem can be regarded as a version of the Fundamental Theorem for line integrals.

FT FOR LINE INTEGRALS — Theorem 2

Let *C* be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$.

Let *f* be a differentiable function of two or three variables whose gradient vector ∇f is continuous on *C*.

Then,

$$\int_C \nabla f \cdot d\mathbf{r} = f\left(\mathbf{r}\left(b\right)\right) - f\left(\mathbf{r}\left(a\right)\right)$$

NOTE

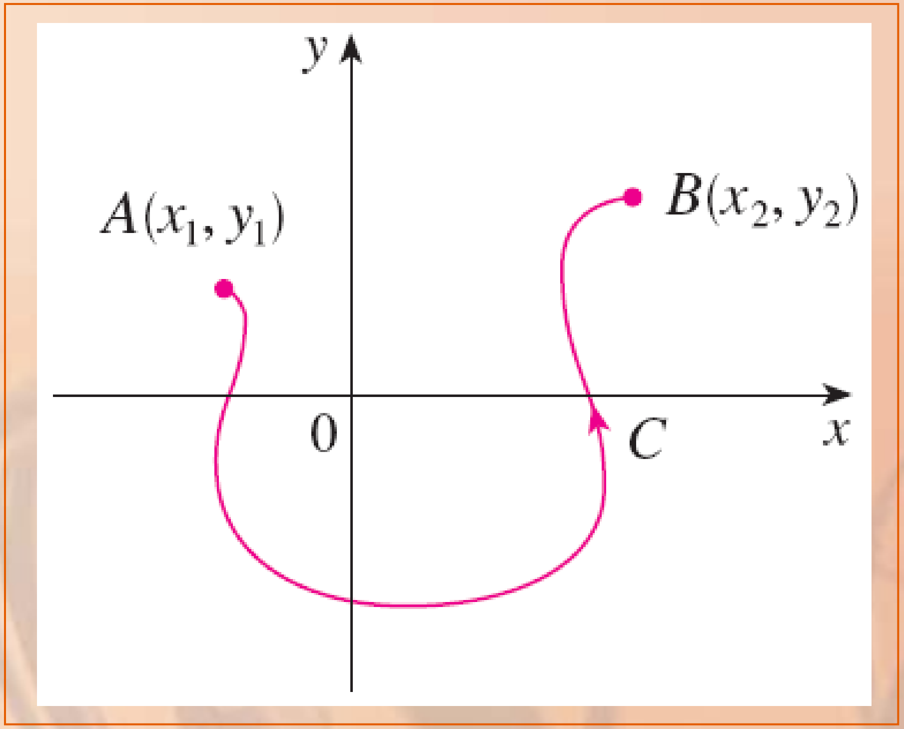
Theorem 2 says that we can evaluate the line integral of a conservative vector field (the gradient vector field of the potential function *f*) simply by knowing the value of *f* at the endpoints of *C*.

- In fact, it says that the line integral of ∇f is the net change in *f*.

NOTE

If f is a function of two variables and C is a plane curve with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$, Theorem 2 becomes:

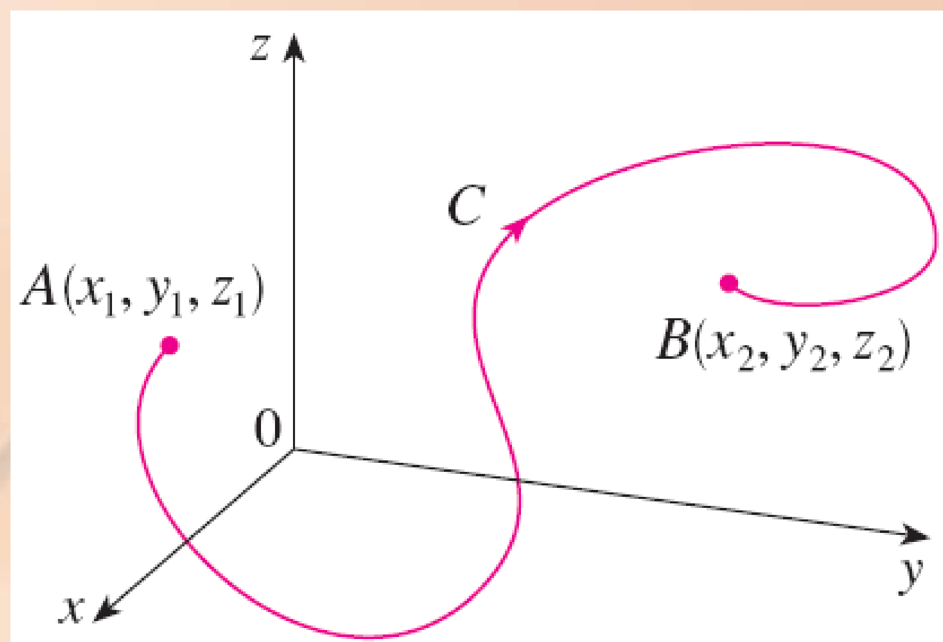
$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$$



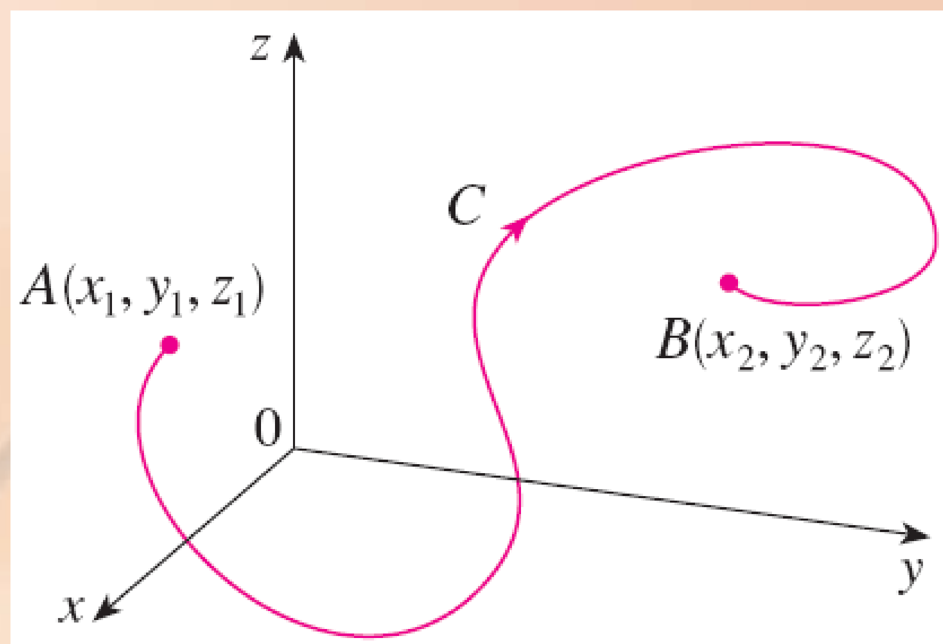
NOTE

If f is a function of three variables and C is a space curve joining the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$, we have:

$$\int_C \nabla f \cdot d\mathbf{r}$$
$$= f(x_2, y_2, z_2)$$
$$- f(x_1, y_1, z_1)$$



Let's prove Theorem 2 for this case.



Using Definition 13 in Section 12.2, we have:

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f\left(\mathbf{r}(t)\right) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt} f\left(\mathbf{r}(t)\right) dt = f\left(\mathbf{r}(b)\right) - f\left(\mathbf{r}(a)\right)$$

- The last step follows from the FTC (Equation 1).

FT FOR LINE INTEGRALS

Though we have proved Theorem 2 for smooth curves, it is also true for piecewise-smooth curves.

- This can be seen by subdividing *C* into a finite number of smooth curves and adding the resulting integrals.

Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

in moving a particle with mass *m* from the point (3, 4, 12) to the point (2, 2, 0) along a piecewise-smooth curve *C*.

- See Example 4 in Section 12.1

From Section 12.1, we know that **F** is a conservative vector field and, in fact, $\mathbf{F} = \nabla f$, where:

$$f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$$

So, by Theorem 2, the work done is:

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} \\ &= f(2,2,0) - f(3,4,12) \\ &= \frac{mMG}{\sqrt{2^2+2^2}} - \frac{mMG}{\sqrt{3^2+4^2+12^2}} \\ &= mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right) \end{aligned}$$

PATHS

Suppose C_1 and C_2 are two piecewise-smooth curves (which are called paths) that have the same initial point *A* and terminal point *B*.

We know from Example 4 in Section 12.2 that, in general,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

CONSERVATIVE VECTOR FIELD

However, one implication of Theorem 2 is that

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \Delta f \cdot d\mathbf{r}$$

whenever ∇f is continuous.

- That is, the line integral of a conservative vector field depends only on the initial point and terminal point of a curve.

INDEPENDENCE OF PATH

In general, if **F** is a continuous vector field with domain *D*, we say that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

for any two paths C_1 and C_2 in *D* that have the same initial and terminal points.

INDEPENDENCE OF PATH

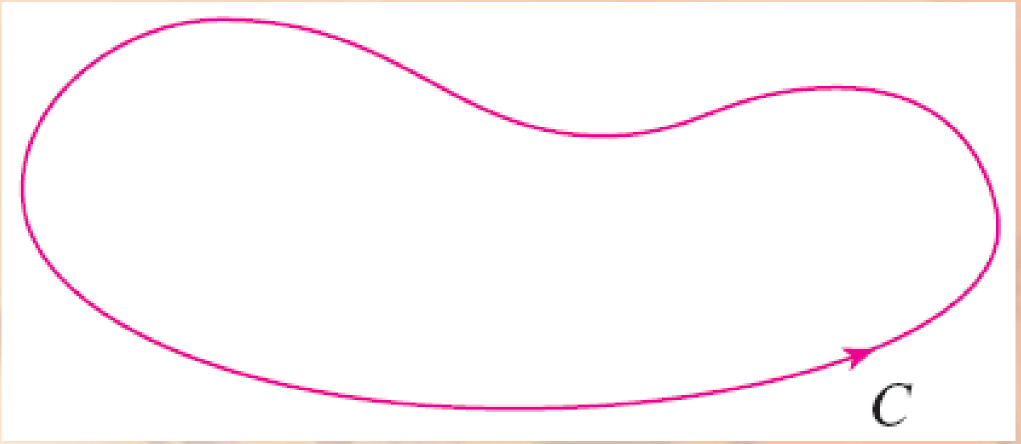
With this terminology, we can say that:

- Line integrals of conservative vector fields are independent of path.

CLOSED CURVE

A curve is called closed if its terminal point coincides with its initial point, that is,

$$\mathbf{r}(b) = \mathbf{r}(a)$$



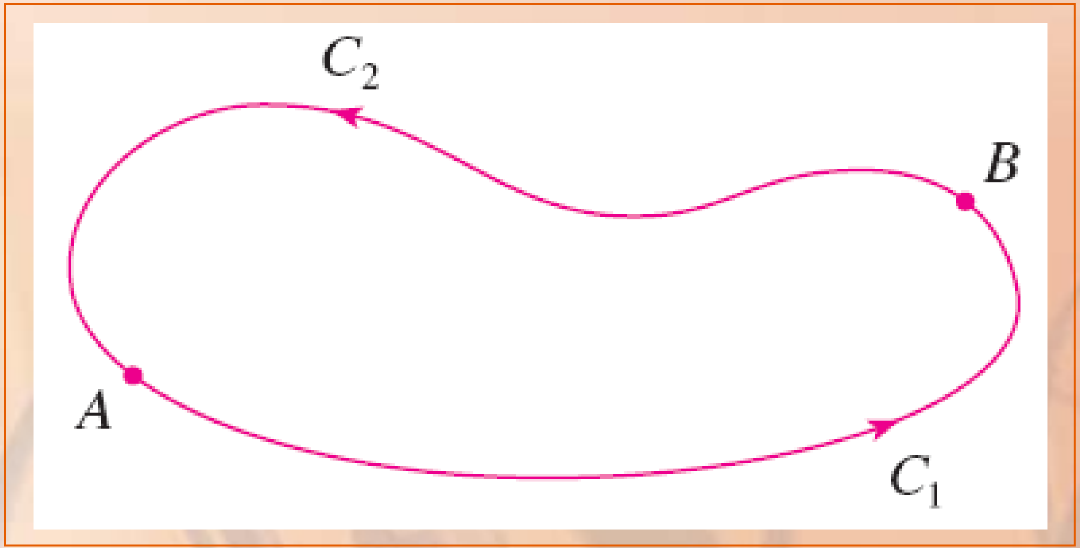
Suppose:

- $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in *D*.
- *C* is any closed path in *D*

INDEPENDENCE OF PATH

Then, we can choose any two points *A* and *B* on *C* and regard *C* as:

- Being composed of the path C_1 from *A* to *B* followed by the path C_2 from *B* to *A*.



Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

- This is because C_1 and $-C_2$ have the same initial and terminal points.

INDEPENDENCE OF PATH

Conversely, if it is true that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ whenever *C* is a closed path in *D*, then we demonstrate independence of path as follows.

INDEPENDENCE OF PATH

Take any two paths C_1 and C_2 from *A* to *B* in *D* and define *C* to be the curve consisting of C_1 followed by $-C_2$.

Then,

$$0 = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Hence, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$

- So, we have proved the following theorem.

$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in *D* if and only if:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

for every closed path *C* in *D*.

INDEPENDENCE OF PATH

We know that the line integral of any conservative vector field **F** is independent of path.

It follows that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path.

PHYSICAL INTERPRETATION

The physical interpretation is that:

- The work done by a conservative force field (such as the gravitational or electric field in Section 13.1) as it moves an object around a closed path is 0.

INDEPENDENCE OF PATH

The following theorem says that the only vector fields that are independent of path are conservative.

- It is stated and proved for plane curves.
- However, there is a similar version for space curves.

INDEPENDENCE OF PATH

We assume that D is open—which means that, for every point P in D, there is a disk with center P that lies entirely in D.

- So, D doesn't contain any of its boundary points.

In addition, we assume that *D* is connected.

- This means that any two points in *D* can be joined by a path that lies in *D*.

CONSERVATIVE VECTOR FIELD **Theorem 4**

Suppose **F** is a vector field that is continuous on an open, connected region *D*.

If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in *D*, then **F** is a conservative vector field on *D.*

- That is, there exists a function *f* such that $\nabla f = \mathbf{F}$

Let *A*(*a*, *b*) be a fixed point in *D*.

We construct the desired potential function *f* by defining

$$f(x, y) = \int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}$$

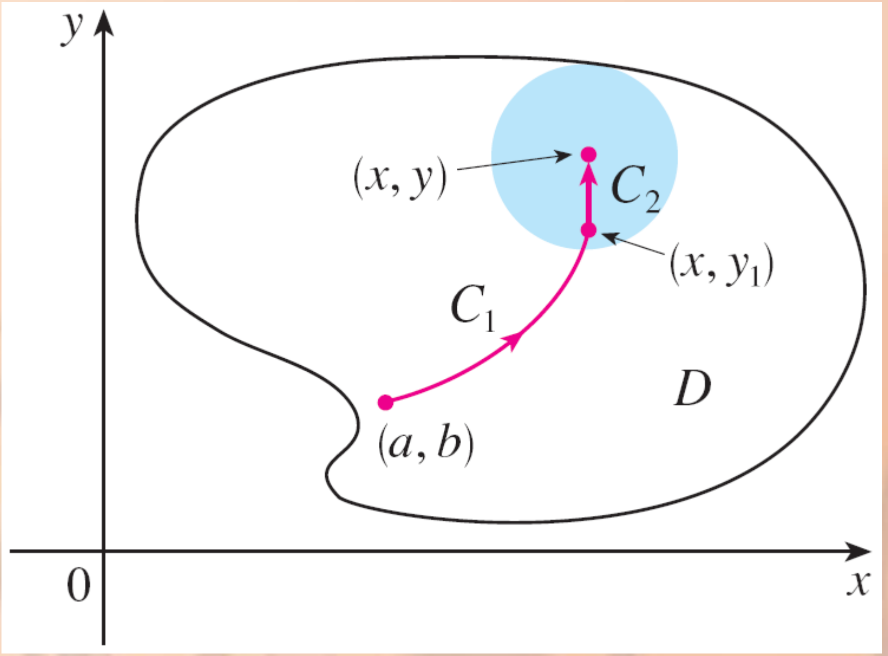
for any point in (*x*, *y*) in *D*.

As $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, it does not matter which path *C* from (*a*, *b*) to (*x*, *y*) is used to evaluate *f*(*x*, *y*).

Since *D* is open, there exists a disk contained in *D* with center (*x*, *y*).

Choose any point (x_1, y) in the disk with $x_1 < x$.

Then, let C consist of any path C_1 from (a, b) to (x_1, y) followed by the horizontal line segment C_2 from (x_1, y) to (x, y).



Then,

$$f(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_{(a,b)}^{(x_1,y)} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

- Notice that the first of these integrals does not depend on *x*.
- Hence,

$$\frac{\partial}{\partial x} f(x, y) = 0 + \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Proof

If we write **F** = *P* **i** +*Q* **j**,

then

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} P\,dx + Q\,dy$$

On C_2, y is constant; so, $dy = 0$.

Using t as the parameter, where $x_1 \le t \le x$, we have:

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \int_{C_2} P\,dx + Q\,dy$$

$$= \frac{\partial}{\partial x} \int_{x_1}^{x} P(t, y)\,dt = P(x, y)$$

by Part 1 of the Fundamental Theorem of Calculus (FTC1).

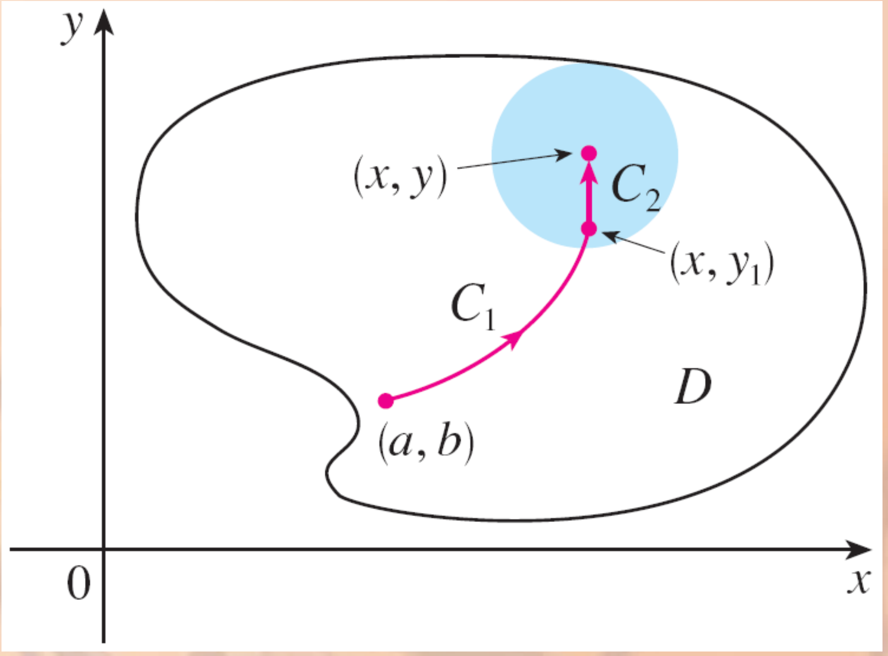
A similar argument, using a vertical line segment, shows that:

$$\frac{\partial}{\partial y} f(x,y)$$

$$= \frac{\partial}{\partial y} \int_{C_2} P\,dx + Q\,dy$$

$$= \frac{\partial}{\partial y} \int_{y_1}^{y} Q(x,t)\,dt$$

$$= Q(x,y)$$



Thus,

$$\begin{aligned} \mathbf{F} &= P\,\mathbf{i} + Q\,\mathbf{j} \\ &= \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} \\ &= \nabla f \end{aligned}$$

- This says that **F** is conservative.

The question remains:

- How is it possible to determine whether or not a vector field is conservative?

Suppose it is known that **F** = *P* **i** + *Q* **j**

is conservative—where *P* and *Q* have

continuous first-order partial derivatives.

- Then, there is a function *f* such that $F = \nabla f$, that is,

$$P = \frac{\partial f}{\partial x} \quad \text{and} \quad Q = \frac{\partial f}{\partial y}$$

DETERMINING CONSERVATIVE VECTOR FIELDS

- Therefore, by Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

 Theorem 5

If

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$$

is a conservative vector field, where *P* and *Q* have continuous first-order partial derivatives on a domain *D*, then, throughout *D*, we have:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

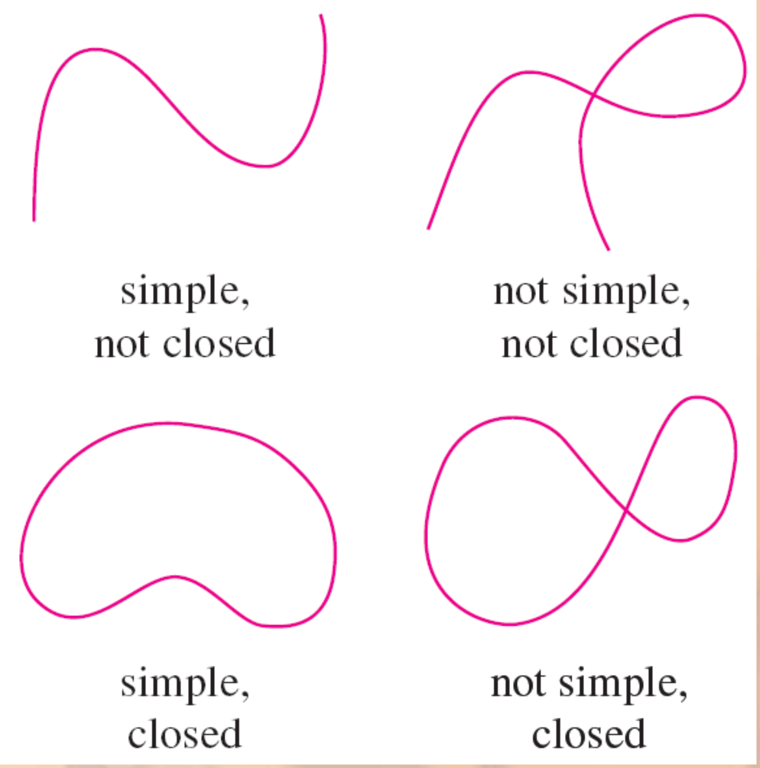
The converse of Theorem 5 is true only for a special type of region.

SIMPLE CURVE

To explain this, we first need the concept of a simple curve—a curve that doesn't intersect itself anywhere between its endpoints.

- **r**(a) = **r**(b) for a simple, closed curve.

- However, $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$.



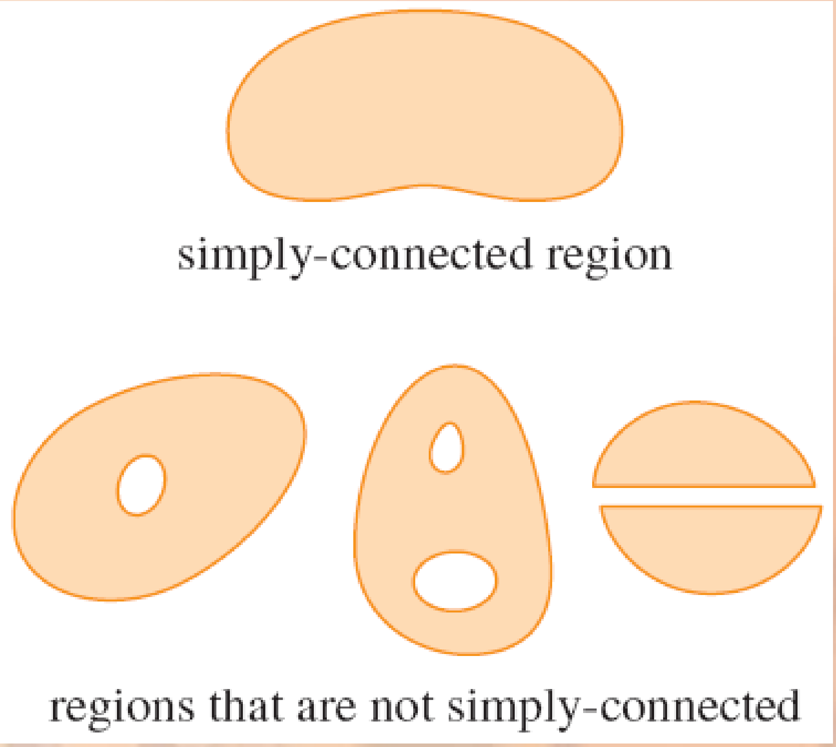
In Theorem 4, we needed an open, connected region.

- For the next theorem, we need a stronger condition.

SIMPLY-CONNECTED REGION

A simply-connected region in the plane is a connected region *D* such that every simple closed curve in *D* encloses only points in *D*.

- Intuitively, it contains no hole and can't consist of two separate pieces.



CONSERVATIVE VECTOR FIELDS

In terms of simply-connected regions, we now state a partial converse to Theorem 5 that gives a convenient method for verifying that a vector field on $°^{2}$ is conservative.

- The proof will be sketched in Section 12.3 as a consequence of Green’s Theorem.

Let **F** = *P* **i** + *Q* **j** be a vector field on an open simply-connected region *D*.

Suppose that *P* and *Q* have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout *D.*

- Then, *F* is conservative.

Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$$

is conservative.

- Let $P(x, y) = x - y$ and $Q(x, y) = x - 2$.
- Then, $\dfrac{\partial P}{\partial y} = -1 \qquad \dfrac{\partial Q}{\partial x} = 1$
- As $\partial P/\partial y \neq \partial Q/\partial x$, **F** is not conservative by Theorem 5.

CONSERVATIVE VECTOR FIELDS

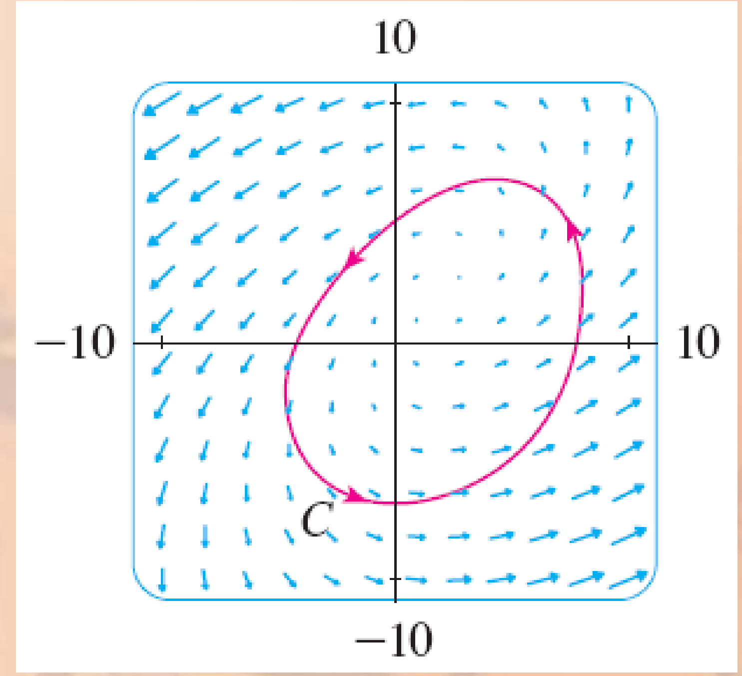
The vectors in the figure that start on the closed curve *C* all appear to point in roughly the same direction as *C*.

- Thus, it looks as if

  $$\int_C \mathbf{F} \cdot dr > 0$$

  and so **F** is not conservative.

- The calculation in Example 2 confirms this impression.



Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

is conservative.

- Let $P(x, y) = 3 + 2xy$ and $Q(x, y) = x^2 - 3y^2$.
- Then, $$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$$

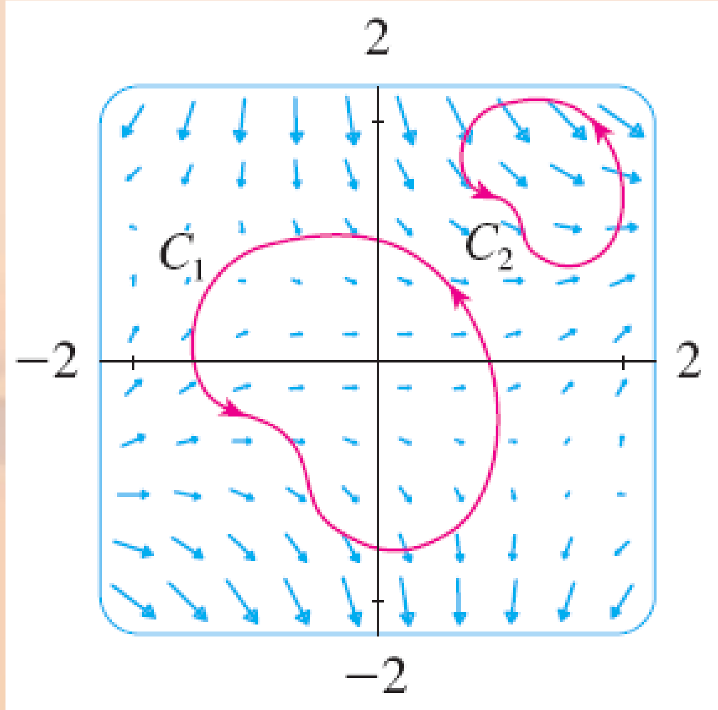
CONSERVATIVE VECTOR FIELDS **Example 3**

- Also, the domain of **F** is the entire plane ($D = \mathbb{R}^2$), which is open and simply-connected.

- Therefore, we can apply Theorem 6 and conclude that **F** is conservative.

CONSERVATIVE VECTOR FIELDS

Some vectors near the curves C_1 and C_2 in the figure point in approximately the same direction as the curves, whereas others point in the opposite direction.



- So, it appears plausible that line integrals around all closed paths are 0.
- Example 3 shows that **F** is indeed conservative.

FINDING POTENTIAL FUNCTION

In Example 3, Theorem 6 told us that **F** is conservative.

However, it did not tell us how to find the (potential) function *f* such that $\mathbf{F} = \nabla f$.

FINDING POTENTIAL FUNCTION

The proof of Theorem 4 gives us a clue as to how to find *f*.

- We use “partial integration” as in the following example.

FINDING POTENTIAL FUNCTION Example 4

a. If $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

b. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} \qquad 0 \le t \le \pi$

From Example 3, we know that **F** is conservative.

So, there exists a function *f* with $\nabla f = \mathbf{F}$, that is,

$$f_x(x, y) = 3 + 2xy$$

$$f_y(x, y) = x^2 - 3y^2$$

Integrating Equation 7 with respect to *x*, we obtain:

$$f(x, y) = 3x + x^2y + g(y)$$

- Notice that the constant of integration is a constant with respect to *x*, that is, a function of *y*, which we have called $g(y)$.

Next, we differentiate both sides of Equation 9 with respect to *y*:

$$f_y(x, y) = x^2 + g'(y)$$

Comparing Equations 8 and 10,

we see that:

$$g'(y) = -3y^2$$

- Integrating with respect to *y*, we have:

$$g(y) = -y^3 + K$$

where *K* is a constant.

Putting this in Equation 9,

we have

$$f(x, y) = 3x + x^2y - y^3 + K$$

as the desired potential function.

To use Theorem 2, all we have to know are the initial and terminal points of *C*, namely,

$$\mathbf{r}(0) = (0, 1)$$

$$\mathbf{r}(\pi) = (0, -e^{\pi})$$

In the expression for $f(x, y)$ in part a, any value of the constant K will do.

- So, let's choose $K = 0$.

Then, we have:

$$\int_C \mathbf{F} \cdot dr = \int_C \nabla f \cdot dr = f\left(0, -e^{\pi}\right) - f\left(0, 1\right)$$

$$= e^{3\pi} - \left(-1\right) = e^{3\pi} + 1$$

- This method is much shorter than the straightforward method for evaluating line integrals that we learned in Section 13.2

CONSERVATIVE VECTOR FIELDS

A criterion for determining whether or not a vector field **F** on ° ^{3} is conservative is given in Section 13.5

FINDING POTENTIAL FUNCTION

Meanwhile, the next example shows that the technique for finding the potential function is much the same as for vector fields on $\circ^{2}$.

FINDING POTENTIAL FUNCTION Example 5

If

$$\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$$

find a function f such that $\nabla f = \mathbf{F}$.

If there is such a function *f*,

then

$$f_x(x, y, z) = y^2$$

$$f_y(x, y, z) = 2xy + e^{3z}$$

$$f_z(x, y, z) = 3ye^{3z}$$

Integrating Equation 11 with respect to x, we get:

$$f(x, y, z) = xy^2 + g(y, z)$$

where $g(y, z)$ is a constant with respect to x.

Then, differentiating Equation 14 with respect to *y*, we have:

$$f_y(x, y, z) = 2xy + g_y(y, z)$$

- Comparison with Equation 12 gives:

$$g_y(y, z) = e^{3z}$$

Thus,

$$g(y, z) = ye^{3z} + h(z)$$

- So, we rewrite Equation 14 as:

$$f(x, y, z) = xy^2 + ye^{3z} + h(z)$$

Finally, differentiating with respect to z and comparing with Equation 13, we obtain:

$$h'(z) = 0$$

- Therefore, $h(z) = K$, a constant.

The desired function is:

$$f(x, y, z) = xy^2 + ye^{3z} + K$$

- It is easily verified that $\nabla f = \mathbf{F}$.

Let's apply the ideas of this chapter to a continuous force field **F** that moves an object along a path *C* given by:

$$\mathbf{r}(t),\ a \le t \le b$$

where:

- $\mathbf{r}(a) = A$ is the initial point of *C*.
- $\mathbf{r}(b) = B$ is the terminal point of *C*.

By Newton's Second Law of Motion,

the force $\mathbf{F}(\mathbf{r}(t))$ at a point on C is related to

the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ by the equation

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

So, the work done by the force on the object is:

$$W$$

$$= \int_C \mathbf{F} \cdot dr$$

$$= \int_a^b \mathbf{F}\left(\mathbf{r}\left(t\right)\right) \cdot \mathbf{r}'\left(t\right) dt$$

$$= \int_a^b m\mathbf{r}''\left(t\right) \cdot \mathbf{r}'\left(t\right) dt$$

CONSERVATION OF ENERGY

$$= \frac{m}{2}\int_a^b \frac{d}{dt}\left[\mathbf{r}'(t)\cdot\mathbf{r}'(t)\right]dt$$

(Th. 3, Sec. 13.2, Formula 4)

$$= \frac{m}{2}\int_a^b \frac{d}{dt}\left|\mathbf{r}'(t)\right|^2 dt = \frac{m}{2}\left[\left|\mathbf{r}'(t)\right|^2\right]_a^b$$

(FTC)

$$= \frac{m}{2}\left(\left|\mathbf{r}'(b)\right|^2 - \left|\mathbf{r}'(a)\right|^2\right)$$

Therefore,

$$W = \tfrac{1}{2} m \left| \mathbf{v}(b) \right|^2 - \tfrac{1}{2} m \left| \mathbf{v}(a) \right|^2$$

where **v** = **r**' is the velocity.

KINETIC ENERGY

The quantity

$$\tfrac{1}{2}m\left|\mathbf{v}(t)\right|^2$$

that is, half the mass times the square of the speed, is called the kinetic energy of the object.

Therefore, we can rewrite Equation 15 as:

$$W = K(B) - K(A)$$

- This says that the work done by the force field along *C* is equal to the change in kinetic energy at the endpoints of *C*.

Now, let's further assume that **F** is a conservative force field.

- That is, we can write $\mathbf{F} = \nabla f$.

POTENTIAL ENERGY

In physics, the potential energy of an object at the point (x, y, z) is defined as:

$$P(x, y, z) = -f(x, y, z)$$

- So, we have $\mathbf{F} = -\nabla P$.

Then, by Theorem 2, we have:

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= -\int_C \nabla P \cdot d\mathbf{r} \\ &= -\left[P\left(\mathbf{r}(b)\right) - P\left(\mathbf{r}(a)\right) \right] \\ &= P(A) - P(B) \end{aligned}$$

Comparing that equation with Equation 16, we see that:

$$P(A) + K(A) = P(B) + K(B)$$

CONSERVATION OF ENERGY

$$P(A) + K(A) = P(B) + K(B)$$

says that:

- If an object moves from one point *A* to another point *B* under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant.

LAW OF CONSERVATION OF ENERGY

This is called the Law of Conservation of Energy.

- It is the reason the vector field is called conservative.