10

VECTOR FUNCTIONS

VECTOR FUNCTIONS

Later in this chapter, we are going to use vector functions to describe the motion of planets and other objects through space.

- Here, we prepare the way by developing the calculus of vector functions.

10.8

Derivatives and Integrals of Vector Functions

In this section, we will learn how to:

Develop the calculus of vector functions.

The derivative **r'** of a vector function is defined in much the same way as for real-valued functions.

Equation 1

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists.

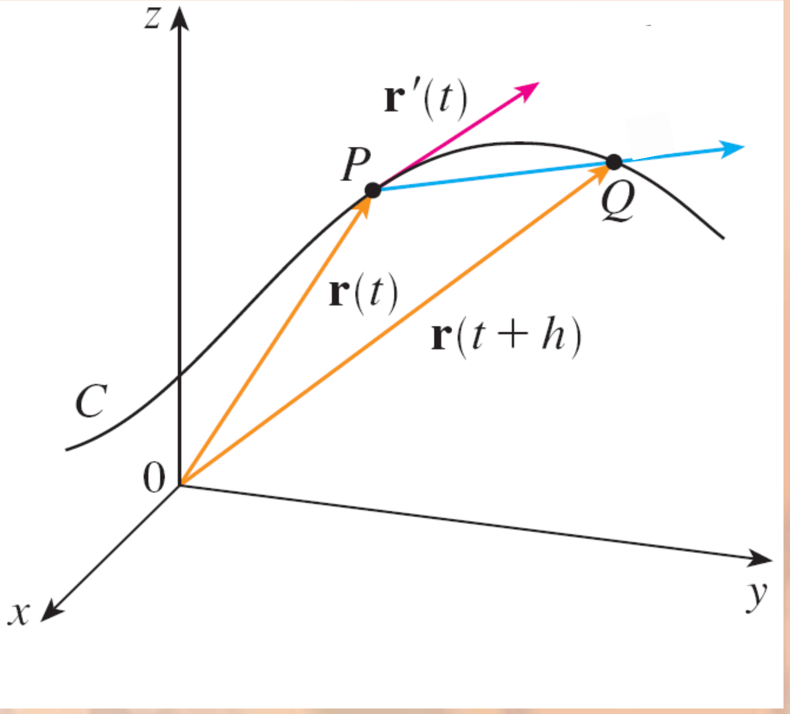
DERIVATIVE

The geometric significance of this definition is shown as follows.

SECANT VECTOR

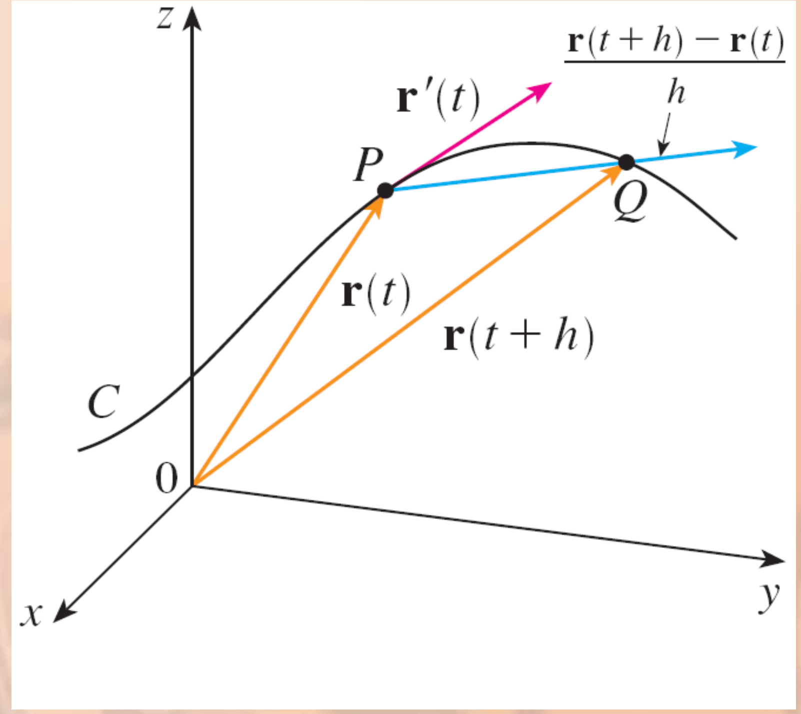
If the points *P* and *Q* have position vectors **r**(*t*) and **r**(*t* + *h*), then $\overrightarrow{PQ}$ represents the vector **r**(*t* + *h*) – **r**(*t*).

- This can therefore be regarded as a secant vector.



If $h > 0$, the scalar multiple $(1/h)(\mathbf{r}(t + h) - \mathbf{r}(t))$ has the same direction as $\mathbf{r}(t + h) - \mathbf{r}(t)$.

- As $h \to 0$, it appears that this vector approaches a vector that lies on the tangent line.

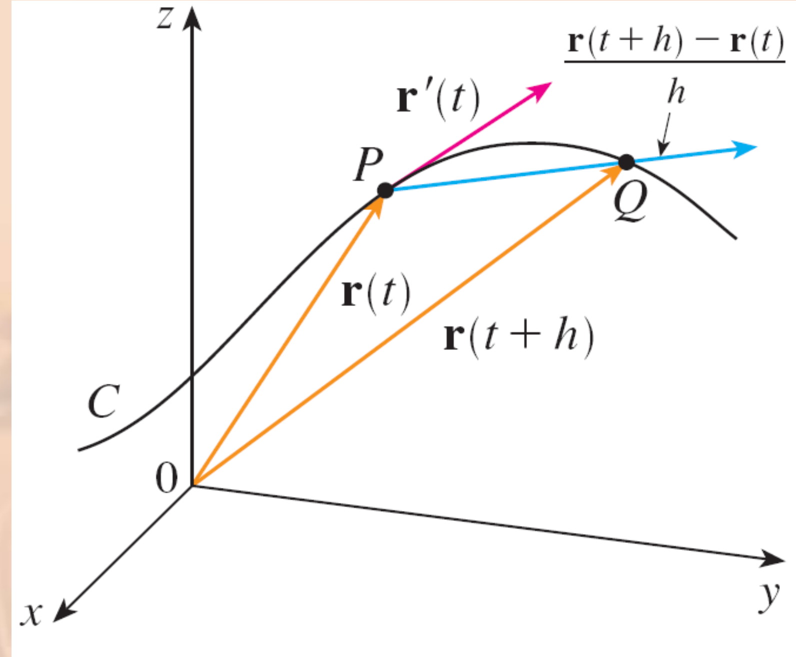


TANGENT VECTOR

For this reason, the vector **r'**(t) is called the tangent vector to the curve defined by **r** at the point P,

provided:

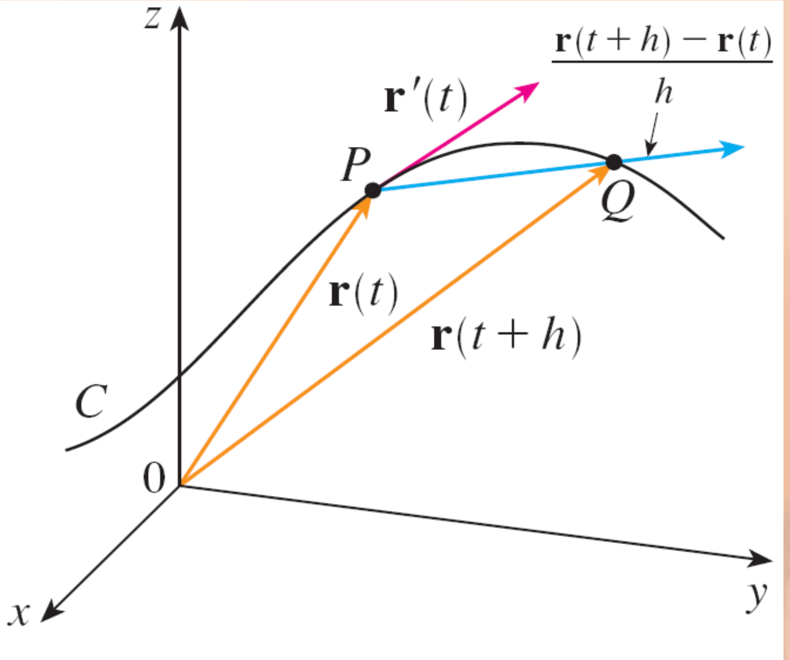
- **r'**(t) exists
- **r'**(t) ≠ **0**



The tangent vector

TANGENT LINE

The tangent line to *C* at *P* is defined to be the line through *P* parallel to the tangent vector **r'**(*t*).



The tangent vector

We will also have occasion to consider the unit tangent vector:

$$T(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

DERIVATIVES

The following theorem gives us a convenient method for computing the derivative of a vector function **r**:

- Just differentiate each component of **r**.

If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then:

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

Proof

$$\mathbf{r}'(t)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\mathbf{r}(t+\Delta t) - \mathbf{r}(t)]$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\langle f(t+\Delta t), g(t+\Delta t), h(t+\Delta t), \rangle - \langle f(t), g(t), h(t) \rangle\right]$$

$$= \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$= \left\langle \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$= \langle f'(t), g'(t), h'(t) \rangle$$

a. Find the derivative of

$$\mathbf{r}(t) = (1 + t^3)\,\mathbf{i} + te^{-t}\,\mathbf{j} + \sin 2t\,\mathbf{k}$$

b. Find the unit tangent vector at the point where $t = 0$.

According to Theorem 2, we differentiate each component of **r**:

$$\mathbf{r}'(t) = 3t^2\,\mathbf{i} + (1 - t)e^{-t}\,\mathbf{j} + 2\cos 2t\,\mathbf{k}$$

As **r**(0) = **i** and **r'**(0) = **j** + 2**k**, the unit tangent vector at the point (1, 0, 0) is:

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1+4}}$$

$$= \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$$

Example 2

For the curve $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (2 - t)\,\mathbf{j}$,

find **r'**(*t*) and sketch the position vector **r**(1)

and the tangent vector **r'**(1).

We have:

$$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} - \mathbf{j}$$

and

$$\mathbf{r}'(1) = \frac{1}{2}\mathbf{i} - \mathbf{j}$$

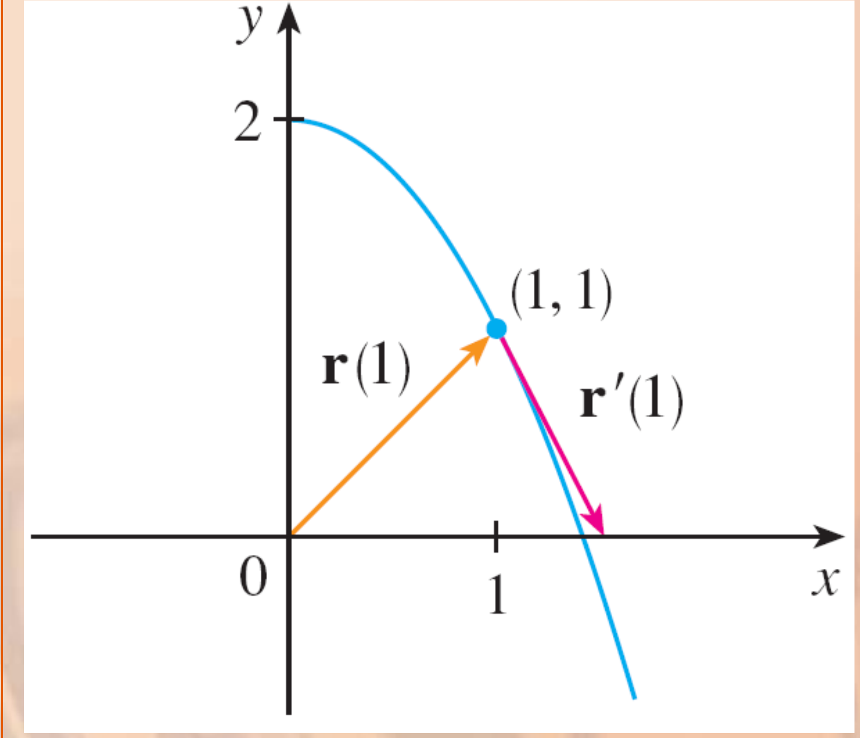
The curve is a plane curve.

Elimination of the parameter from the equations $x = \sqrt{t}$, $y = 2 - t$ gives:

$$y = 2 - x^2, \quad x \geq 0$$

The position vector **r**(1) = **i** + **j** starts at the origin.

The tangent vector **r'**(1) starts at the corresponding point (1, 1).



Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $(0, 1, \pi/2)$.

The vector equation of the helix is:

$$\mathbf{r}(t) = \langle 2 \cos t, \sin t, t \rangle$$

Thus,

$$\mathbf{r}'(t) = \langle -2 \sin t, \cos t, 1 \rangle$$

The parameter value corresponding to the point $(0, 1, \pi/2)$ is $t = \pi/2$.

- So, the tangent vector there is:

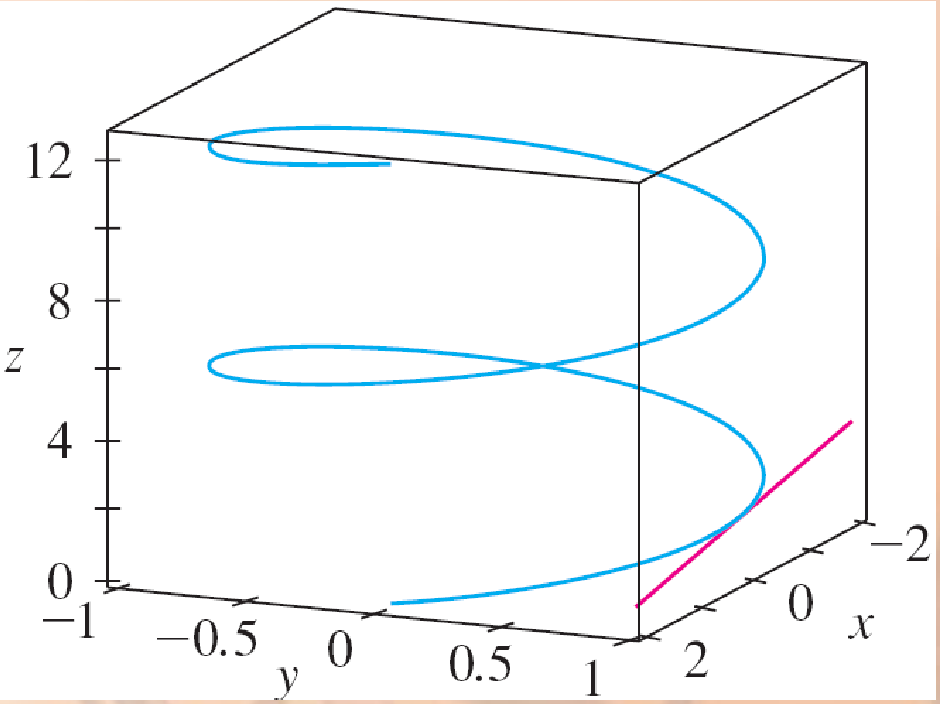
$$\mathbf{r}'(\pi/2) = \langle -2, 0, 1 \rangle$$

The tangent line is the line through $(0, 1, \pi/2)$ parallel to the vector $\langle -2, 0, 1 \rangle$.

- So, by Equations 2 in Section 10.5, its parametric equations are:

$$x = -2t \qquad y = 1 \qquad z = \frac{\pi}{2} + t$$

The helix and the tangent line in the Example 3 are shown.



SECOND DERIVATIVE

Just as for real-valued functions,

the second derivative of a vector function **r**

is the derivative of **r'**, that is, **r''** = (**r'**)'.

- For instance, the second derivative of the function in Example 3 is:

$$\mathbf{r}''(t) = \langle -2\cos t, \sin t, 0 \rangle$$

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-valued functions.

Suppose:

- **u** and **v** are differentiable vector functions
- *c* is a scalar
- *f* is a real-valued function

Then,

$$1.\ \frac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)]=\mathbf{u}'(t)+\mathbf{v}'(t)$$

$$2.\ \frac{d}{dt}[c\mathbf{u}(t)]=c\mathbf{u}'(t)$$

$$3.\ \frac{d}{dt}[f(t)\mathbf{u}(t)]=f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$$

4. $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$

5. $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$

6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'\left(f(t)\right)$ (Chain Rule)

This theorem can be proved either:

- Directly from Definition 1
- By using Theorem 2 and the corresponding differentiation formulas for real-valued functions

The proof of Formula 4 follows.

- The remaining are left as exercises.

Let

$$\mathbf{u}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$$

$$\mathbf{v}(t) = \langle g_1(t), g_2(t), g_3(t) \rangle$$

- Then, $\mathbf{u}(t) \cdot \mathbf{v}(t)$

$$= f_1(t)g_1(t) + f_2(t)g_2(t) + f_3(t)g_3(t)$$

$$= \sum_{i=1}^{3} f_i(t)g_i(t)$$

- So, the ordinary Product Rule gives:

$$\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \frac{d}{dt}\sum_{i=1}^{3} f_i(t)g_i(t)$$

$$= \sum_{i=1}^{3}\frac{d}{dt}[f_i(t)g_i(t)]$$

$$= \sum_{i=1}^{3}[f_i'(t)g_i(t) + f_i(t)g_i'(t)]$$

$$= \sum_{i=1}^{3} f_i'(t)g_i(t) + \sum_{i=1}^{3} f_i(t)g_i'(t)$$

$$= \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$$

Example 4

Show that, if |**r**(*t*)| = *c* (a constant), then **r'**(*t*) is orthogonal to **r**(*t*) for all *t*.

Since

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$$

and c^2 is a constant,

Formula 4 of Theorem 3 gives:

$$\begin{aligned} 0 &= \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] \\ &= \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) \\ &= 2\mathbf{r}'(t) \cdot \mathbf{r}(t) \end{aligned}$$

Thus,

$$\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$$

- This says that **r'**(*t*) is orthogonal to **r**(*t*).

Geometrically, this result says:

- If a curve lies on a sphere with center the origin, then the tangent vector **r'**(t) is always perpendicular to the position vector **r**(t).

The definite integral of a continuous vector function **r**(*t*) can be defined in much the same way as for real-valued functions—except that the integral is a vector.

However, then, we can express the integral of **r** in terms of the integrals of its component functions *f*, *g*, and *h* as follows.

INTEGRALS

$$\int_a^b \mathbf{r}(t)\,dt$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{r}(t_i^*)\,\Delta t$$

$$= \lim_{n\to\infty} \left[\left(\sum_{i=1}^{n} f(t_i^*)\,\Delta t \right)\mathbf{i} + \left(\sum_{i=1}^{n} g(t_i^*)\,\Delta t \right)\mathbf{j} + \left(\sum_{i=1}^{n} h(t_i^*)\,\Delta t \right)\mathbf{k} \right]$$

Thus,

$$\int_a^b \mathbf{r}(t)\,dt$$

$$= \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}$$

- This means that we can evaluate an integral of a vector function by integrating each component function.

We can extend the Fundamental Theorem of Calculus to continuous vector functions:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

- Here, **R** is an antiderivative of **r**, that is, **R'**(t) = **r**(t).
- We use the notation $\int$ **r**(t) dt for indefinite integrals (antiderivatives).

If **r**(*t*) = 2 cos *t* **i** + sin *t* **j** + 2*t* **k**, then

$$\int \mathbf{r}(t)\,dt = \left(\int 2\cos t\,dt\right)\mathbf{i} + \left(\int \sin t\,dt\right)\mathbf{j} + \left(\int 2t\,dt\right)\mathbf{k}$$

$$= 2\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + t^2\mathbf{k} + \mathbf{C}$$

where:

- **C** is a vector constant of integration
- $\int_0^{\pi/2} \mathbf{r}(t)\,dt = [2\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + t^2\mathbf{k}]_0^{\pi/2}$

$$= 2\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\mathbf{k}$$